Some Applications of Algebraic System Solving

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Firstly, I would like to thank the organizers of TIME-2010 and the steering committee for inviting me to give this plenary talk.

It is a great pleasure for me to have this opportunity to talk to you today

☺

I would like my talk not to be a monologue,
so I invite you to ask or express your opinions at any moment…

I wouldn't like my talk to look like this:

- I was born in 1962, and when I became 10 year old, the educational system in Spain changed and we were moved to a "Modern Mathematics" environment.
- Primary school teachers had to face teaching students up to 14 year old (instead of up to 10 year old), and many, suspicious regarding "modern mathematics" added (on their own) "old style" ugly looking text books to the "modern mathematics" books.

- These books were anything but didactically oriented.
- I was really afraid of the "ponds problems" (swimming pool was a too modern word) like:
  two sources providing … liters per second and … liters per second (respectively) fill a pond of … cubic meters. How long will it take to fill the pond?
- They were solved using "rules of three" (even applied to problems with more than 3 numerical values), that I never mastered.
- The key problem (apart from ugly details like units appearing only at the very last step) is that the mathematical theory wasn't known by the student, so it was all rather bungle …
  Each different problem seemed to need a different approach.

- So the main reason for the student's (my) failure was the lack of a theoretical basis for the problem.
- All problems disappeared when my father taught me an equational approach to this sort of problems (i.e., as I was taught the <u>needed</u> theory!).
- I was really surprised by the fact that I COULD THEN SOLVE ALL PROBLEMS THAT COULD BE EXPRESSED AS A LINEAR EQUATION OR LINEAR SYSTEM!!!

- Now returning to 2010, most CAS (like DERIVE) include the possibility to SOLVE ALGEBRAIC SYSTEMS (using Gröbner Bases, what is usually kept hidden to the user, as Gauss' method when solving linear systems).

- This opens a new world of possibilities, as happened to me with linear system solving!!! ANY ("reasonably" sized) PROBLEM that can be expressed in terms of an ALGEBRAIC SYSTEM can then be SOLVED!!!

- Why is using a CAS a key issue?

- Firstly, because **it can handle non-assigned variables**:

(x+y)^2 - (x-y)^2 = 4xy

DIF(SIN(x^2),x) = 2·x·COS(x^2)

(these computations are of a qualitative higher complexity level than those performed by "usual" computer languages).

- Secondly, because **exact arithmetic** is a must for relying on the correctness of the computations (otherwise we would have to perform numerical analysis computations regarding errors).

- **Show DERIVE Example (digitos_aprox.dfw).**

How many digits are correct?
Eugenio Roanes-Lozano
TIME 2010 (7-VII-2010)
Created: 1-VII-2010

```
#1:   x := 1.000001

                                    2
#2:             ITERATE(APPROX(x := x , 6), i, 1, 30) = x := 1

#3:   x := 1.000001

                           2
#4:   ITERATE(APPROX(x := x , 16), i, 1, 30) = x :=

                                                                      ~
        2088892037531073768478669480691219817320136012776524630349058935~

                                                                      ~
        1839050919558903697002137718037424015508171334853449442756817951~

                                                                      ~
        7199955392756059526580396995899103101364913387302360400011404597~

                                                                      ~
        8255303006942505540378607384264776359762400092946173857241699401~

                                                                      ~
        0365505727773707863258378504481403183543057674896121959328475368~

                                                                      ~
        2142344605345137814697796714081396654687333478956873816661397684~

                                                                      ~
        4672846507076770550595712610133157620529940124277781042548470965~

        1407902604158631936

#5:   x := 1.000001

                           2
#6:   ITERATE(APPROX(x := x , 32), i, 1, 30) = x :=

                                                                      ~
        2088892025819431696035903453396348230115243233425706856278178975~

                                                                      ~
        6420195159782665017569304753399453432958875158824230895379079081~

                                                                      ~
        0494596627742250302955969554365841295627378123125714809466905178~
```

~
482851238083183112219151100791180252550106389376358239415854133 4~
~
160645643453182046892621923656214580756425612681965647695087111 3~
~
536069858440949633907345292075387931637736022080728519964725081 6~
~
531643223171597614756192597451288357115013546570285470359027397 3~

2655152337899279681

Only the first 8 digits coincide in this case (precision 16 vs 32).

References:

E. Roanes-Lozano: Precisión Indefinida y Matemática Elemental. *Bol. Soc. “Puig Adam” de Profs. de Matemáticas* 31 (1992) 33-52.

AN ELEMENTARY INTRODUCTION TO ALGEBRAIC SYSTEMS

- Let us recall some details of my Keynote Lecture at TIME'2004.
- Approximate methods were the only alternative until the 60's, when the first **general and effective method was found: Gröbner bases**.
- Although implementations of GB are incorporated to all CAS, they are not widely known.
- DERIVE's **SOLVE** command for polynomial systems internally calculates a GB if the system isn't linear.

SOLVING LINEAR SYSTEMS (MULTIVARIATE CASE)

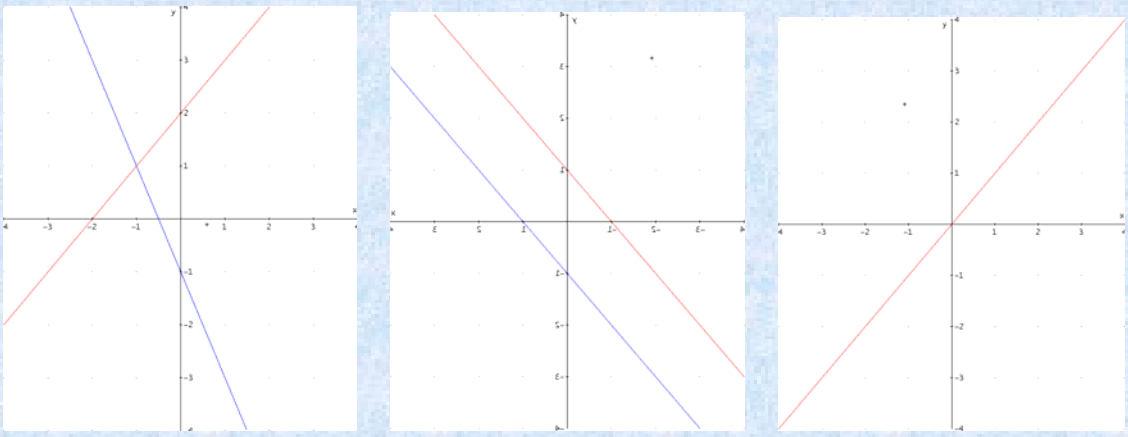
#13: SOLVE([y = x + 2, y = - 2·x - 1], [x, y])

we look for the values of *x* and *y* that satisfy all equations simultaneously:

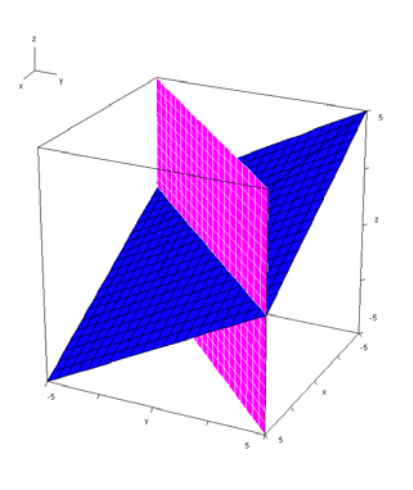
#14: [x = -1 ∧ y = 1]

We are asking for the intersection of the solution sets corresponding to each equation (lines in 2D, planes in 3D and hyperplanes if >3D)

- .Linear systems can also have a unique solution, no solution or infinite solutions:



- In case the dimension of the space is greater than 2, the solution set of the linear system can be a linear variety of intermediate dimension.



- Then,all we can do is to express it in a simpler way!

ALGEBRAIC SYSTEMS

- An *algebraic equation* is an equation of the form:

general polynomial = 0

- *An algebraic system*, also called *polynomial system*, is a set of algebraic equations.

SOLVING AN ALGEBRAIC SYSTEM

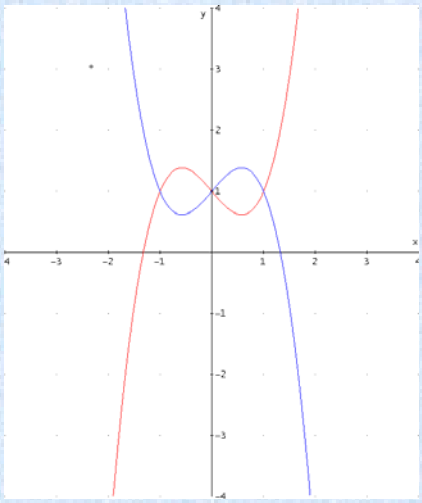
- What happens when we ask DERIVE to SOLVE an algebraic system?

#36: SOLVE([$x^2 - 1 = y$, $-x^2 - x + 2 = y$], [x, y])

#37: $[x = 1 \wedge y = 0, x = -\frac{3}{2} \wedge y = \frac{5}{4}]$

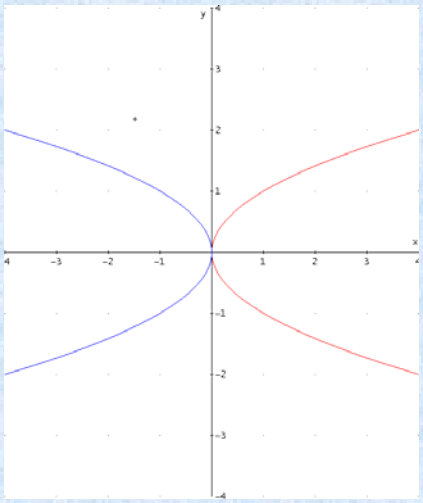
- We are asking DERIVE to look for the values of *x* and *y* that satisfy all the equations simultaneously.

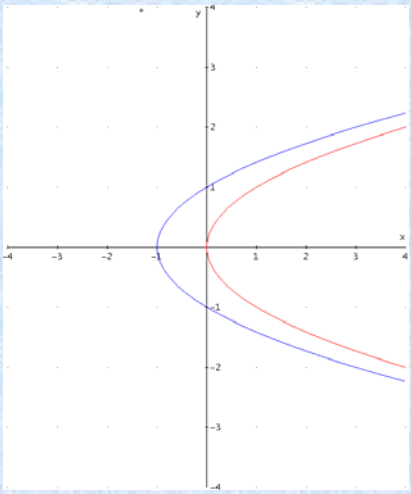
- From the geometrical point of view, we are asking for the intersection of the solution sets corresponding to the equations (curves in 2D; surfaces in 3D or hypersurfaces in >3D):

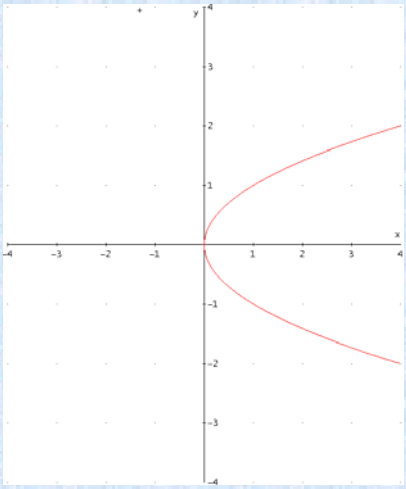


- The solution set of an algebraic system is denoted *algebraic variety*, and it can consist of unconnected components.

- Agebraic systems with a unique solution, no solution or infinite solutions exist.



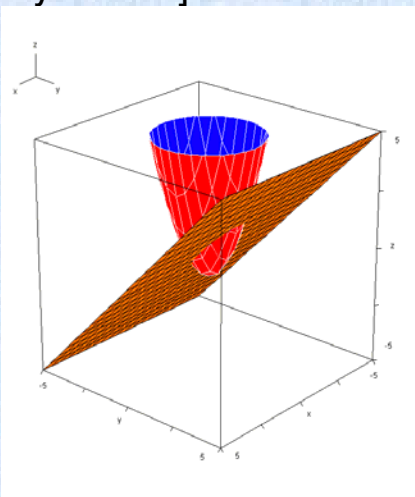




- And If the dimension of the space is > 2, the solution set of the polynomial system could be an algebraic variety of intermediate dimension:

#55: SOLVE([$x^2 + y^2 - z - 1 = 0$, $y - z = 0$], [x,y,z])

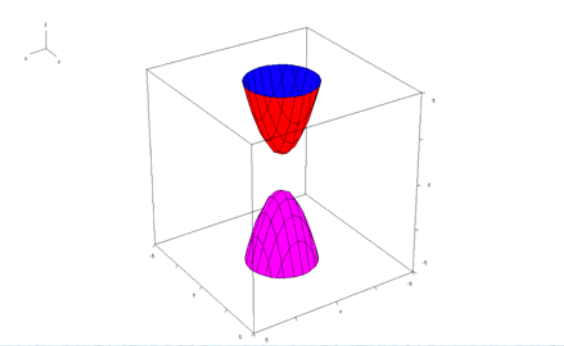
#56: [$x^2 + y^2 - z = 1 \wedge y - z = 0$]

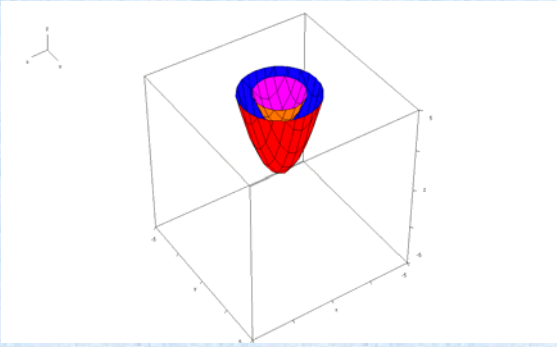


- $\mathbb{R}$ vs $\mathbb{C}$: whether the system has solutions or not also depends on the set where we are looking for such solutions. There are no real solutions for:

#57: $[x^2 + y^2 - z + 1 = 0, -x^2 - y^2 - z - 1 = 0]$

#58: $[x^2 + y^2 - z = 0, x^2 + y^2 - z + 3 = 0]$





- But the first system has complex solutions!:

#59: SOLVE($[x^2 + y^2 - z + 1 = 0, -x^2 - y^2 - z - 1 = 0]$, [x, y])

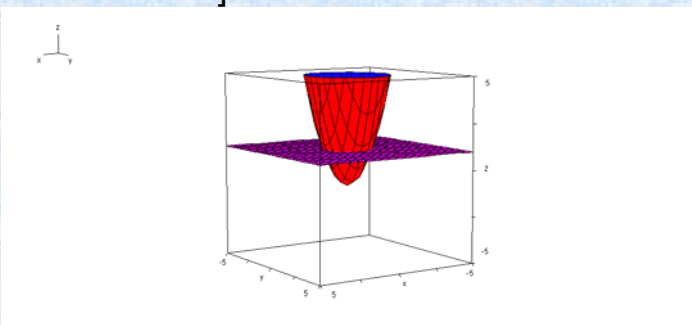
#60: $[x^2 + y^2 - z = -1 \wedge x^2 + y^2 = -1]$ ($x = i$, $y = 0$, $z = 0$ satisfy both)

GB FINDS THE LOWEST DEGREE (SIMPLEST) EXPRESSION

(a canonical basis of any given ideal, that characterizes it (once the variable ordering and term ordering are chosen).

#65: SOLVE($[x^2 + y^2 - z - 1 = 0, z - 1 = 0]$, [x, y, z])

#66: $[x^2 + y^2 = 2 \wedge z = 1]$



calls GB, that expresses the ideal as an intersection of a vertical cylinder and a horizontal plane:

#67: GROEBNER_BASIS($[x^2 + y^2 - z - 1, z - 1]$, [x, y, z])

#68: $[z - 1, x^2 + y^2 - 2]$

- A WHOLE NEW WORLD OF POSSIBILITIES OPENS TO US!!!

References:

E. Roanes-Lozano, E. Roanes-Macías, L. M. Laita: The Geometry of Algebraic Systems and Their Exact Solving Using Groebner Bases. *Computing in Science and Engineering* 6/2 (2004) 76-79.

E. Roanes-Lozano, E. Roanes-Macías, L. M. Laita: Some Applications of Gröbner Bases. *Computing in Science and Engineering* 6/3 (2004) 56-60.

E. Roanes-Lozano: Pictures at a DERIVE'S exhibition (interpreting DERIVE's SOLVE command). Keynote Lecture at TIME-2004 conference. Published in: J. Böhm (ed.): *Proceedings of TIME 2004* (CD-ROM). bk teachware Schriftenreihe, Nr. SR-41, 2004.

- **Show DERIVE Example (3_color_grafos.dfw)**

Graph 3-coluring (D. Bayer's idea)
Eugenio Roanes-Lozano
TIME 2010 (7-VII-2010)
Created: 3-VII-2010

THE PROBLEM: Can we decide when a given map is 3-colourable?

(This is NOT the famous "4-colour problem": all maps with connected countries can be coloured with at most 4 colours in a way such that countries that share a border that is not only a point are coloured with different colours).

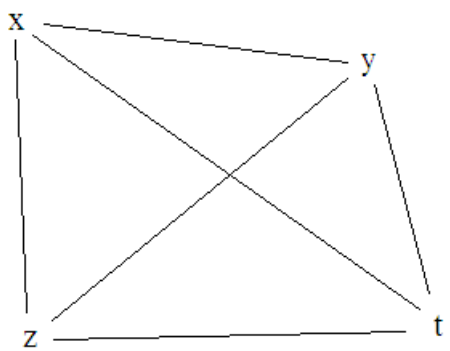
Coding:

The countries are the vertices of the graph.

There is an edge between two nodes iff the two countries share a border.

Vertices are denoted by polynomial variables. For example: x,y,z,t

A graph is introduced as a vector of vectors. For example:



#1: $G1 := \begin{bmatrix} x & y \\ y & z \\ z & x \\ t & x \\ t & y \\ t & z \end{bmatrix}$

The 3 colours are designated by the 3 cubic roots of unit:

#2: $\mathrm{SOLVE}(x^3 - 1 = 0, x) = \left(x = -\frac{1}{2} - \frac{\sqrt{3}\cdot i}{2} \vee x = -\frac{1}{2} + \frac{\sqrt{3}\cdot i}{2} \vee x = 1\right)$

As a colour is assigned to each vertex, for each vertex, x, x^3-1=0 must hold.

#3: $\mathrm{pol\_vertex}(\mathrm{var}) := \mathrm{var}^3 - 1$

#4: $\mathrm{pol\_vertex}(x) = x^3 - 1$

So, given a graph, we can consider the polynomials corresponding to the vertices:

#5: $\mathrm{MAP\_LIST}(\mathrm{pol\_vertex}(x), x, \mathrm{VARIABLES}(G1)) = \left[x^3 - 1, y^3 - 1, z^3 - 1, t^3 - 1\right]$

Edges are denoted by polinomials of the form x^2+xy+y^2. The idea comes from the fact that x^3-y^3=1-1=0 and:

#6: $\mathrm{FACTOR}(x^3 - y^3) = (x - y)\cdot(x^2 + x\cdot y + y^2)$

(if one factor doesn't vanish, the other should).

#7: $\mathrm{pol\_edge}(e) := e_1^2 + e_1\cdot e_2 + e_2^2$

Given a graph, we can then consider the polynomials corresponding to edges:

#8: $\mathrm{MAP\_LIST}(\mathrm{pol\_edge}(e), e, G1) = \left[x^2 + x\cdot y + y^2, y^2 + y\cdot z + z^2, x^2 + x\cdot z + z^2, x^2 + t\cdot x + t^2, y^2 + t\cdot y + t^2, z^2 + t\cdot z + t^2\right]$

And we can put them together:

#9: $\mathrm{APPEND}(\mathrm{MAP\_LIST}(\mathrm{pol\_vertex}(x), x, \mathrm{VARIABLES}(G1)), \mathrm{MAP\_LIST}(\mathrm{pol\_edge}(e), e, G1)) = \left[x^3 - 1, y^3 - 1, z^3 - 1, t^3 - 1, x^2 + x\cdot y + y^2, y^2 + y\cdot z + z^2, x^2 + x\cdot z + z^2, x^2 + t\cdot x + t^2, y^2 + t\cdot y + t^2, z^2 + t\cdot z + t^2\right]$

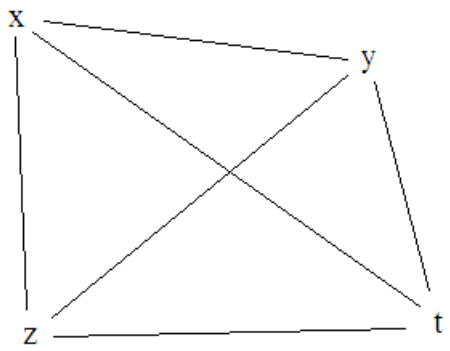
or even create a function that, given the graph, looks for its 3-colourings:

```
#10:  three_col(G) := SOLVE(APPEND(MAP_LIST(pol_vertex(x), x,
        VARIABLES(G)), MAP_LIST(pol_edge(e), e, G)), VARIABLES(G))
```

This function can be used to create a Boolean function that checks wether the graph is 3-colourable or not:

```
#11:  is_3_col(G) := ¬ three_col(G) = []
```

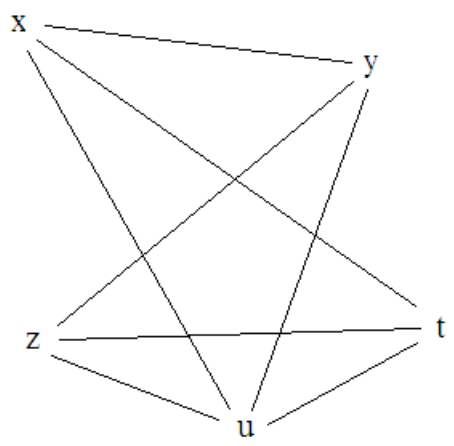
Example 1: there is no possible 3-colouring for G1




```
#12:                          is_3_col(G1) = false
```

Example 2: there is at least one possible 3-colouring for G2

#13: $G2 := \begin{bmatrix} x & y \\ y & z \\ z & t \\ t & x \\ u & x \\ u & y \\ u & z \\ u & t \end{bmatrix}$



#14: `is_3_col(G2) = true`

Moreover: solving the system gives us the possible 3-colourings:

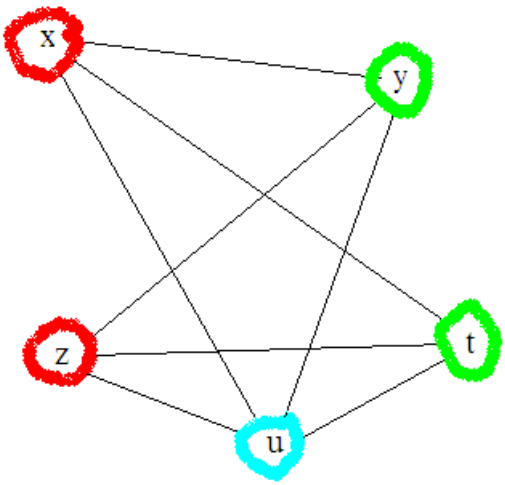
#15: $$\text{three\_col}(G2) = \left[x = 1 \wedge y = -\frac{1}{2} + \frac{\sqrt{3}\cdot i}{2} \wedge z = 1 \wedge t = -\frac{1}{2} + \frac{\sqrt{3}\cdot i}{2} \wedge u = -\frac{1}{2} - \frac{\sqrt{3}\cdot i}{2},\ x = 1 \wedge y = -\frac{1}{2} - \frac{\sqrt{3}\cdot i}{2} \wedge z = 1 \wedge t = -\frac{1}{2} - \frac{\sqrt{3}\cdot i}{2} \wedge u = -\frac{1}{2} + \frac{\sqrt{3}\cdot i}{2},\ x = -\frac{1}{2} + \frac{\sqrt{3}\cdot i}{2} \wedge y = 1 \wedge z = -\frac{1}{2} + \frac{\sqrt{3}\cdot i}{2} \wedge t = 1 \wedge u = -\frac{1}{2} - \frac{\sqrt{3}\cdot i}{2},\ x = -\frac{1}{2} + \frac{\sqrt{3}\cdot i}{2} \wedge y = -\frac{1}{2} - \frac{\sqrt{3}\cdot i}{2} \wedge z = -\frac{1}{2} + \frac{\sqrt{3}\cdot i}{2} \wedge t = -\frac{1}{2} - \frac{\sqrt{3}\cdot i}{2} \wedge u = 1,\ x = -\frac{1}{2} - \frac{\sqrt{3}\cdot i}{2} \wedge y = 1 \wedge z = -\frac{1}{2} - \frac{\sqrt{3}\cdot i}{2} \wedge t = 1 \wedge u = -\frac{1}{2} + \frac{\sqrt{3}\cdot i}{2},\ x = -\frac{1}{2} - \frac{\sqrt{3}\cdot i}{2} \wedge y = -\frac{1}{2} + \frac{\sqrt{3}\cdot i}{2} \wedge z = -\frac{1}{2} - \frac{\sqrt{3}\cdot i}{2} \wedge t = -\frac{1}{2} + \frac{\sqrt{3}\cdot i}{2} \wedge u = 1\right]$$

Let us consider the first solution:

#16: $$(\text{three\_col}(G2))_1 = \left(x = 1 \wedge y = -\frac{1}{2} + \frac{\sqrt{3}\cdot i}{2} \wedge z = 1 \wedge t = -\frac{1}{2}\right.$$

$$+ \frac{\sqrt{3} \cdot i}{2} \wedge u = - \frac{1}{2} - \frac{\sqrt{3} \cdot i}{2} \Big)$$

that, if : 1 = red ; -1/2+sqrt(3)i/2 = green ; -1/2-sqrt(3)i/2 = blue , is the 3-colouring:



So algebraic system solving has made it possible to decide wether any given graph is 3-colourable or not (through a smart coding of the graph)!!!

References:

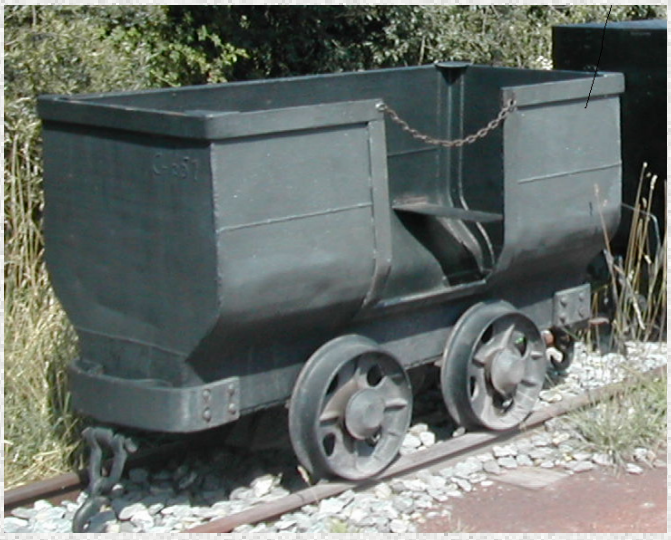
D.A. Bayer, “The Division Algorithm and the Hilbert Scheme”, Ph.D. Thesis, Harvard University, 1982.

William W. Adams, Philippe Loustaunau: An Introduction to Grobner Bases. AMS Graduate Studies in Mathematics, 1994 (pages 102-105).

- Railways:

(RENFE high speed 100 series)

are guided transportation systems:



Turnout: switch in the direct line position

To trail trough a switch set against.

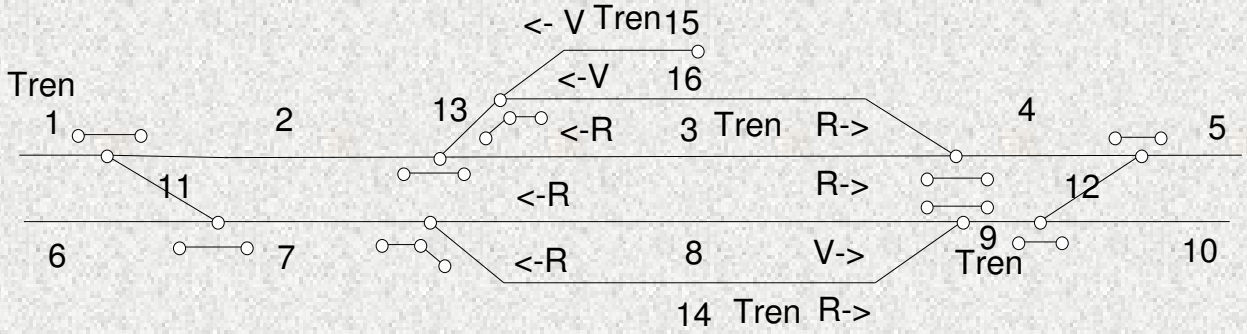
What is a railway interlocking?

- Device that avoids combinations of signals and switches that could lead in the worst case to a collision or trailing.

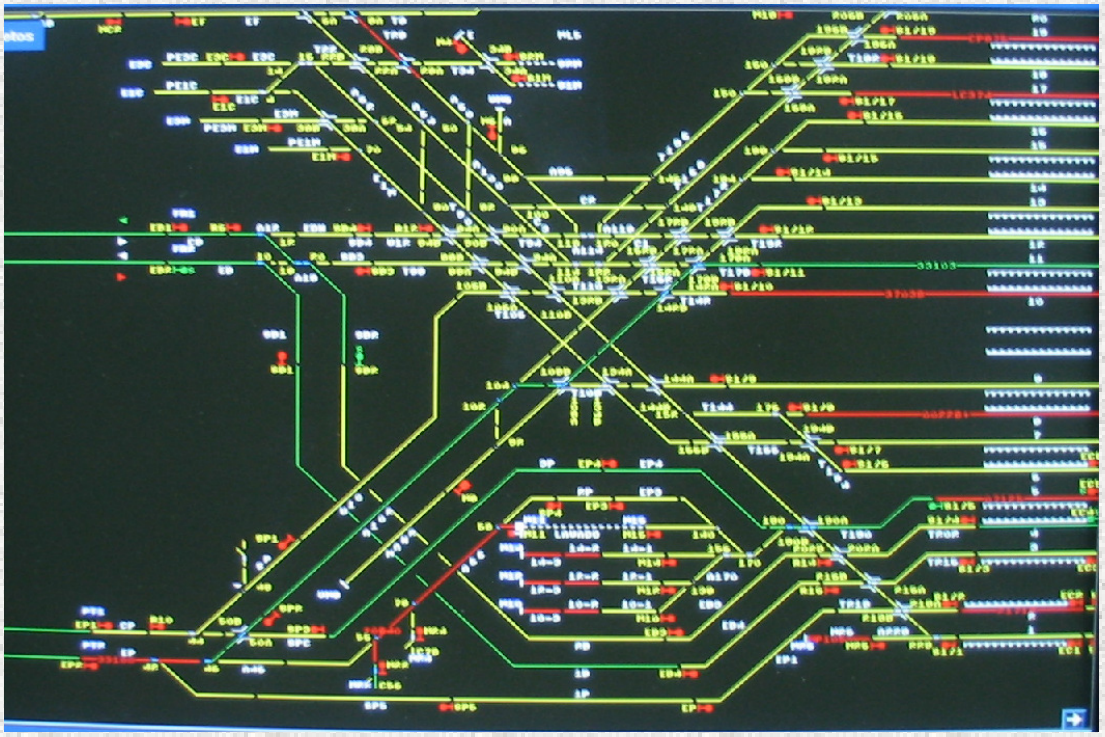
- It is not trivial!

Small station (trailing allowed):

1 -> 1,2,3 ; 14 ->14 ; 9 -> 9,10,8 ; 3 -> 3 ; 15 ->15 (trailing: 13,16,2,1,3)



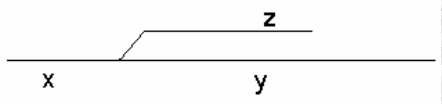
- Imaging the problem at a big station:



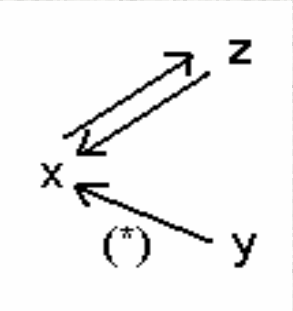
A small part of the relays of a relay-based railway interlocking system of a big railway station

Our approach:

Interpretation as an oriented graph:



the corresponding digraph is:



(*) if trailing through a switch set against is allowed.

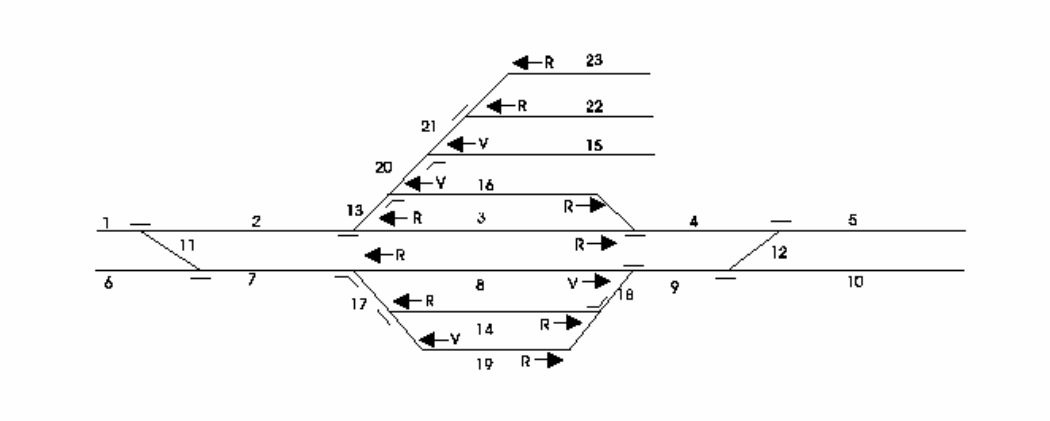
- Pass from one section to an adjacent one must be:

- topologically possible

- allowed by the position of the switches of the turnouts

- allowed by the colour of the semaphores.

- But the transitive closure of the digraph should be considered:



- And also the reflexive closure.

- The proposed situation is translated into an algebraic system (equations of degree ≤ 2).

- Curiously enough, the safetyness of the proposed situation is equivalent to the compatibility of the system.

<u>Coding.-</u> An idea of the author: system *SIST* summarizes the info from the semaphore and switches digraph.

Sections are represented by variables and trains by positive integers.

a) Equation:

$$variable\text{-}num=0$$

is included in *SIST* ifff train number *num* is in section *variable.*

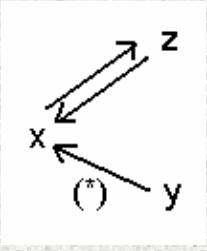
- For example: $x\text{-}11=0$

b) Equation

$$variable1 \cdot (variable1 - variable2) = 0$$

*i*s included in *SIST* iff sections *variable1 and variable2 are* adjacent and it is possible to pass from section *variable1* to section *variable2 according to the* position of the switches and obeying the semaphores.

- For example,



is represented including in *SIST the equations*:

$$x \cdot (x-z) = 0 \quad ; \quad z \cdot (z-x) = 0 \quad ; \quad y \cdot (y-x) = 0 \ .$$

<u>Remark.-</u> We suppose that there is at most one train per section. Consequently, there can't be two equations such as

$$variable\text{-}núm1 = 0 \quad ; \quad variable\text{-}núm2 = 0$$
$$(núm1 \neq núm2).$$

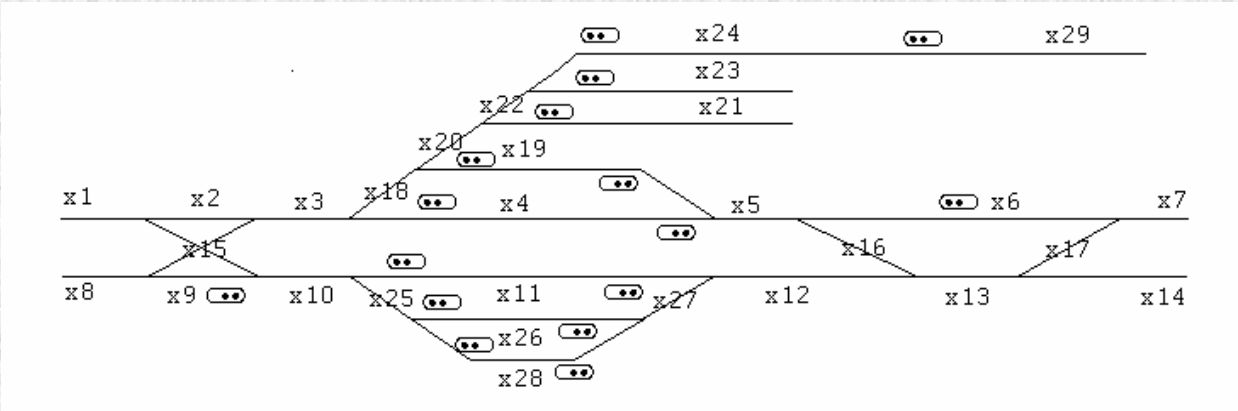
in *SIST*.

<u>Proposition.-</u> A section is accesible by more than one train iff *SIST* is incompatible.

Justification.- Train *2* in section *x*: $x-2=0$ in *SIST*.
If passing from *x* to *y* is allowed: $x\cdot(x-y)=0$ is in *SIST*.
But then: $x=2 \Rightarrow y=2$.
And if it is possible to pass from section *y* to section *m*: $y\cdot(y-m)=0$ is in *SIST*.
Consequently: $m=2$.
Intuitively, value 2 propagates through all the sections accesible from *x*, not just to the adjacent ones..

Consecuence.- The proposed position of switches and semaphores can be authorized iff *SIST* is compatible.

Example:



References:
E. Roanes-Lozano, E. Roanes-Macías, L.M. Laita: Railway Interlocking Systems and Groebner Bases. *Mathematics and Computers in Simulation* 51/5 (2000) 473-481.

Show: encla1ba(GB).mws
Verbally: idea of curve contained in surface
Show: recta_euler.mws

Railway interlockings with Maple (using GB)

Eugenio Roanes-Lozano

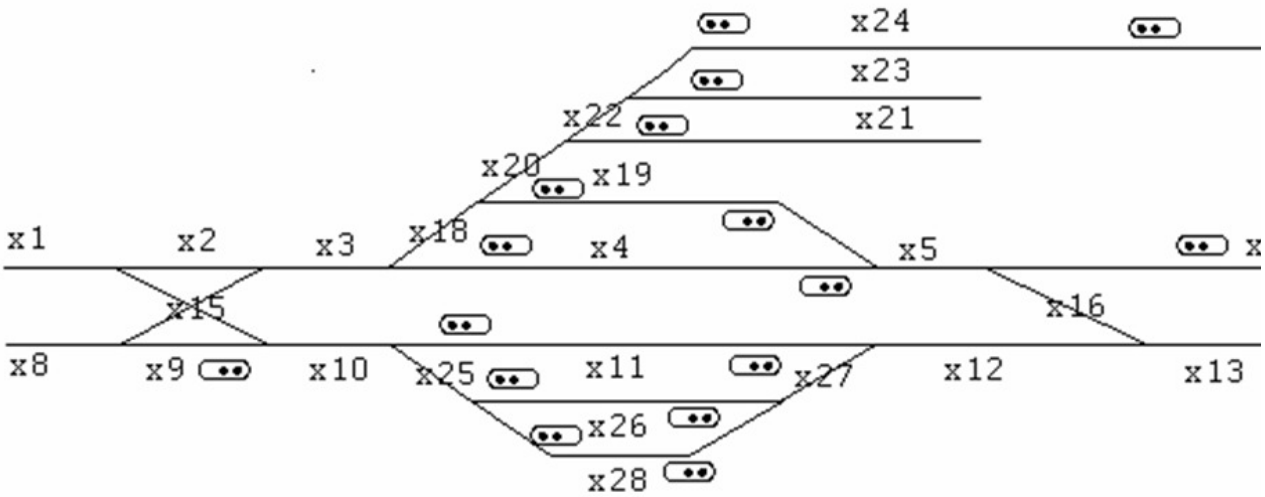
The GB package and our interlocking package have to be loaded first:

```
> restart;
> with(grobner):
Warning, grobner is deprecated. Please, use Groebner.
> read(`d:/congres/2010/TIME2010-Plenaria/Transparencias/trensolv.mpl`):
```

EXAMPLE:

Global variables inicialization:

```
> inicializa();
```



Sections:

```
> LV_:=['x||i' $ i=1..29];
```

$$LV\_ := [x1, x2, x3, x4, x5, x6, x7, x8, x9, x10, x11, x12, x13, x14, x15, x16, x17, x18, x19, x20, x21, x22, x23, x24, x25, x26, x27, x28, x29]$$

Adjacent sections:

```
> adyacente(x24,x29);
> GD_;
```

$$\{x24\,(x24 - x29), x29\,(x24 - x29)\}$$

Position of the switches of the turnouts:

```
> desvio(x7,x6,x17,0);
> desvio(x13,x14,x17,0);
```

```
> desvio(x5,x6,x16,0);
> desvio(x13,x12,x16,0);
> desvio(x5,x4,x19,1);
> desvio(x12,x11,x27,0);
> desvio(x27,x28,x26,0);
> desvio(x25,x28,x26,0);
> desvio(x10,x11,x25,0);
> desvio(x10,x9,x15,0);
> desvio(x8,x9,x15,0);
> desvio(x1,x2,x15,0);
> desvio(x3,x2,x15,0);
> desvio(x3,x4,x18,0);
> desvio(x18,x20,x19,1);
> desvio(x20,x22,x21,1);
> desvio(x22,x24,x23,0);
> GD_;
```

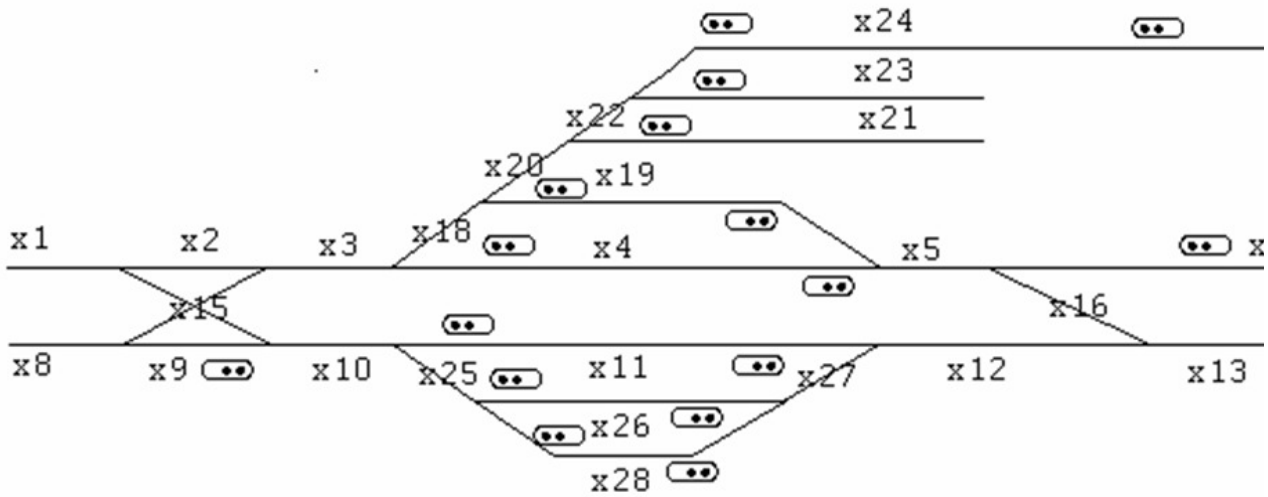
$$\{x1\,(x1-x2), x10\,(x10-x11), x10\,(x10-x9), x11\,(x10-x11), x11\,(x12-x11), x12\,(x12-x11), x12\,(x13-x12), x13\,(x13-x12), x13\,(x13-x14), x14\,(x13-x14), x15\,(x1-x15), x15\,(x10-x15), x15\,(x3-x15), x15\,(x8-x15), x16\,(x13-x16), x16\,(x5-x16), x17\,(x13-x17), x17\,(x7-x17), x18\,(x18-x19), x18\,(x3-x18), x19\,(x18-x19), x19\,(x5-x19), x2\,(x1-x2), x2\,(x3-x2), x20\,(x18-x20), x20\,(x20-x21), x21\,(x20-x21), x22\,(x20-x22), x22\,(x22-x24), x23\,(x22-x23), x24\,(x22-x24), x24\,(x24-x29), x25\,(x10-x25), x25\,(x25-x28), x26\,(x25-x26), x26\,(x27-x26), x27\,(x12-x27), x27\,(x27-x28), x28\,(x25-x28), x28\,(x27-x28), x29\,(x24-x29), x3\,(x3-x2), x3\,(x3-x4), x4\,(x3-x4), x4\,(x5-x4), x5\,(x5-x19), x5\,(x5-x6), x6\,(x5-x6), x6\,(x7-x6), x7\,(x7-x6), x8\,(x8-x9), x9\,(x10-x9), x9\,(x8-x9)\}$$

Colour of the semaphores:

```
> semaforo(x6,x5,1);
> semaforo(x29,x24,1);
> semaforo(x24,x22,1);
> semaforo(x23,x22,0);
> semaforo(x21,x20,1);
```

```
> semaforo(x19,x18,0);
> semaforo(x19,x5,0);
> semaforo(x4,x3,0);
> semaforo(x4,x5,0);
> semaforo(x11,x10,0);
> semaforo(x11,x12,0);
> semaforo(x26,x25,0);
> semaforo(x26,x27,0);
> semaforo(x28,x25,0);
> semaforo(x28,x27,0);
> semaforo(x9,x10,0);
> GS_;
```

$$\{x11\,(x10-x11), x11\,(x11-x10), x11\,(x11-x12), x11\,(x12-x11), x19\,(x18-x19), x19\,(x19-x18), x19\,(x19-x5), x19\,(x5-x19), x23\,(x22-x23), x23\,(x23-x22), x26\,(x25-x26), x26\,(x26-x25), x26\,(x26-x27), x26\,(x27-x26), x28\,(x25-x28), x28\,(x27-x28), x28\,(x28-x25), x28\,(x28-x27), x4\,(x3-x4), x4\,(x4-x3), x4\,(x4-x5), x4\,(x5-x4), x9\,(x10-x9), x9\,(x9-x10)\}$$



Position of the trains:

```
> tren(8,x29,1);
> tren(1,x7,1);
> tren(10,x23,1);
> tren(42,x19,1);
> tren(15,x3,1);
> tren(7,x11,1);
> tren(62,x26,1);
> PT_;
```

$$\{x11-7, x19-42, x23-10, x26-62, x29-8, x3-15, x7-1\}$$

The system to be solved is:

```
> (GD_ union PT_) minus GS_;
```

$$\{x1\,(x1-x2), x10\,(x10-x11), x10\,(x10-x9), x12\,(x12-x11), x12\,(x13-x12), x13\,(x13-x12), x13\,(x13-x14), x14\,(x13-x14), x15\,(x1-x15), x15\,(x10-x15), x15\,(x3-x15), x15\,(x8-x15), x16\,(x13-x16), x16\,(x5-x16), x17\,(x13-x17), x17\,(x7-x17), x18\,(x18-x19), x18\,(x3-x18), x2\,(x1-x2), x2\,(x3-x2), x20\,(x18-x20), x20\,(x20-x21), x21\,(x20-x21), x22\,(x20-x22), x22\,(x22-x24), x24\,(x22-x24), x24\,(x24-x29), x25\,(x10-x25), x25\,(x25-x28), x27\,(x12-x27), x27\,(x27-x28), x29\,(x24-x29), x3\,(x3-x2), x3\,(x3-x4), x5\,(x5-x19), x5\,(x5-x6), x6\,(x5-x6), x6\,(x7-x6), x7\,(x7-x6), x8\,(x8-x9), x9\,(x8-x9), x11-7, x19-42, x23-10, x26-62, x29-8, x3-15, x7-1\}$$

```
> #solve( (GD_ union PT_) minus GS_ , convert(LV_,set) );    #(no sol.)
```

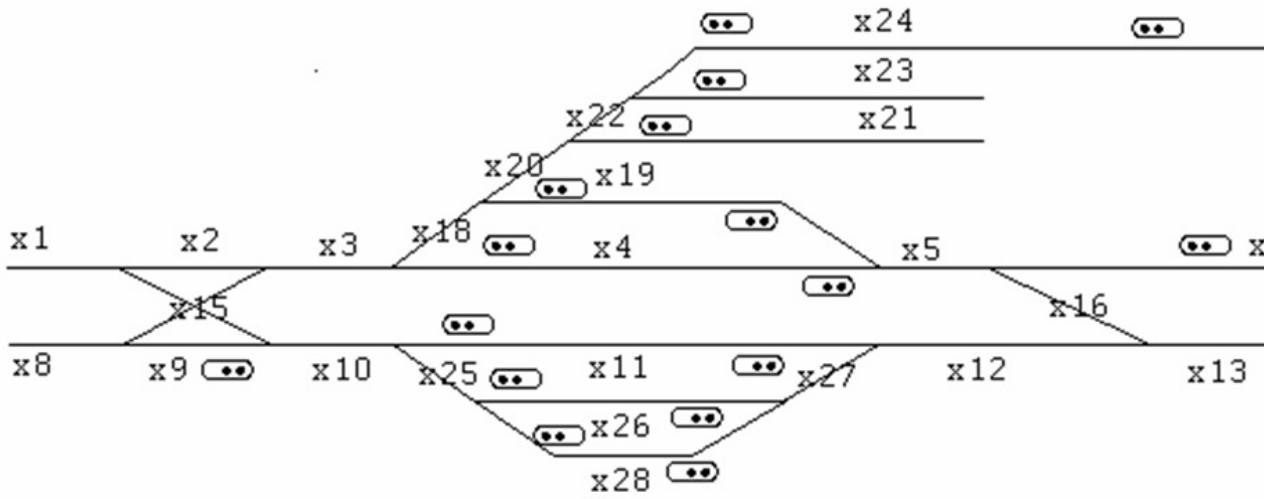
The situation can be analyzed:

```
> tiempo:=time():
> esSegura();
> time()-tiempo;
```

$$false$$

$$0.310$$

Let us check what is happening.
Train 1, departing from section x7 can stay at x7, go to x6, from there to x5 and from there to x19, where train 42 is!
Train 8, departing from section x29 could reach (trailing some switches set against) section x3, occupied by train 15.



Thes situation would be safe turning red (for instance) the semaphores controlling the movements from x6 to x5 and from x24 to x22:

```
> semaforo(x6,x5,0);
> semaforo(x24,x22,0);
```

Let us analyse the new situation:

```
> #solve( (GD_ union PT_) minus GS_ , convert(LV_,set) );      #(hay sol.)
> tiempo:=time();
> esSegura();
> time()-tiempo;
```

$$tiempo := 64.542$$

$$true$$

$$0.151$$

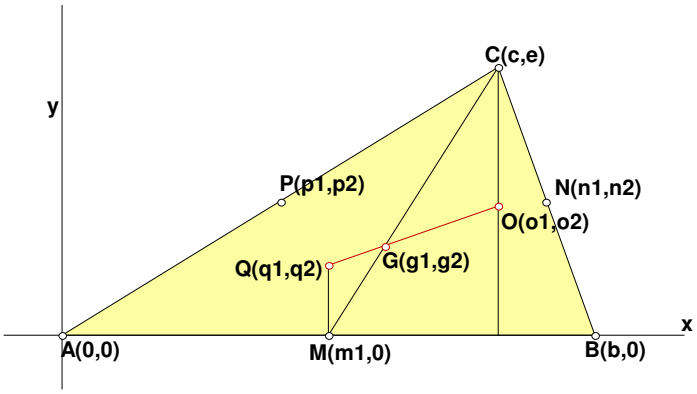
END

```
>
```

RECTA DE EULER (GB method)

Eugenio Roanes-Macías & Eugenio Roanes-Lozano

Theorem: The circumcenter, Q, baricenter, G, and orthocenter, O, of a triangle, ABC, are aligned, being QO=3*QG (vectors).



Points considered (M,N,P are midpoints of sides):

A(0,0), B(b,0), C(c,e), M(m1,0), N(n1,n2), P(p1,p2), Q(q1,q2), G(g1,g2), O(o1,o2)

Parameters: b,c,e

Variables: m1,n1,n2,p1,p2,q1,q2,g1,g2,o1,o2

```
> restart:
```

Hyipothesis and thesis polynomials:

```
> h1:=2*m1-b:                #M is the midpoint of AB
> h2:=2*n1-(b+c):            #N is the midpoint of BC
> h3:=2*n2-e:                #N is the midpoint of BC
> h4:=2*p1-c:                #P is the midpoint of CA
> h5:=2*p2-e:                #P is the midpoint of CA
> h6:=q1-m1:                 #QM is perpendicular to AB
> h7:=c*(q1-p1)+e*(q2-p2): #QP is perpendicular to AC
> h8:=n2*g1-n1*g2:           #G,A,N are collinear
> h9:=p2*(g1-b)+(b-p1)*g2: #G,B,P are colinear
> h10:=o1*(c-b)+o2*e:        #OA is perpendicular to BC
> h11:=c*(o1-b)+e*o2:        #OB is perpendicular to CA
> t1:=(g2-o2)*(q1-o1)-(g1-o1)*(q2-o2): #Q,G,O are colineales
> t2:=(o1-q1)-3*(g1-q1):               #QO=3*QG
> t3:=(o2-q2)-3*(g2-q2):               #QO=3*QG
```

Equal GBs method

```
> hip:=[h1,h2,h3,h4,h5,h6,h7,h8,h9,h10,h11]:
> var:=m1,n1,n2,p1,p2,q1,q2,g1,g2,o1,o2:
> with(Groebner):
> GB1:=Basis( hip , plex(var) ):
> hip_tes:=[h1,h2,h3,h4,h5,h6,h7,h8,h9,h10,h11,t1,t2,t3]:
> GB2:=Basis(hip_tes,plex( var ) ):
> evalb(GB1=GB2);
```

true

GBS equal to <1> method

```
> #Basis( [h1,h2,h3,h4,h5,h6,h7,h8,h9,h10,h11,1-z*t1] , plex(var) );
> #Basis( [h1,h2,h3,h4,h5,h6,h7,h8,h9,h10,h11,1-z*t2] , plex(var) );
> #Basis( [h1,h2,h3,h4,h5,h6,h7,h8,h9,h10,h11,1-z*t3] , plex(var) );
```

References:

Eugenio Roanes-Macías, Eugenio Roanes-Lozano: *Nuevas Tecnologías en Geometría*. Ed. Complutense, Madrid, 1994.

END

```
>
```