Teaching Calculus and Numerical Analysis using CAS according to Bologna Process

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Outline

Introduction

Context:

Why?	The Bologna process has raised the point that learning should be centered on the student rather than the teacher.
How?	A methodology based on problems leads students towards *learning by doing*.
Where?	A Department of Applied Math comprises the wide range of skills needed to address *real* problems.
What?	Models in science and engineering require mathematical tools, from both (*theoretical*) Calculus and (*practical*) Numerical Analysis.

Outline

The society is changing

New skills are required in engineering:

- Leadership.
- Collaborative work.
- Communication.
- ... But scientific knowledge is still necessary.

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- Online learning.
- Interactive and cooperative learning.
- Independent study.
- New environments: laboratories, projects, ...
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Really?

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- Bureaucratic work.
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The Bologna Process

Background

- 1998 Sorbonne Declaration (France, Germany, Italy, UK).

  . . . Europe is not only that of the Euro, of the banks and the economy: it must be a Europe of knowledge as well.

  European Higher Education Area (EHEA) is first mentioned.

- 1999 Bologna Declaration.

  We must in particular look at the objective of increasing the international competitiveness of the European system of higher education. [. . .] We need to ensure that the European higher education system acquires a world-wide degree of attraction equal to our extraordinary cultural and scientific traditions.

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The Bologna Process

Create an European Higher Education Area (EHEA):

- Readable and comparable degrees.
- Undergraduate and postgraduate levels.
- ECTS[a]-compatible credit systems.
- Include lifelong learning activities.
- European dimension in quality assurance.
- Free mobility of students and teachers.

[a] European Credit Transfer System

The Bologna Process

Administrative issues:

- Degrees.
- Levels.
- Credits.
- Mobility.
- No explicit reference to pedagogical methodologies...
- ... but the emphasis on quality is a suggestion that some teaching strategies should be improved.

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The Bologna Process

The EHEA *suggests* renovating methodologies:

Teacher centered

- In Spanish Universities, teaching has often (*uniquely?*) been based upon lectures.
- The current credit measures the teacher's work: 1 credit = 10 class hours.

Student centered

- Active learning requires doing projects, using technology support, ...
- ECTS credits measure the student's work: 1 credit = 25 work hours.

The Bologna Process

The EHEA *suggests* renovating syllabus:

- The undergraduate degree should provide *initial* training.
- General (transversal) competencies, rather than exhaustive knowledge, should be acquired at this level.
- The University degree should be integrated into lifelong learning.

Outline

History of Calculus

Calculus has been formalized:

- Initially, techniques were algorithmic and applied: Newton, Leibniz, Euler, Gauss.
- The formal foundations of Mathematical Analysis are established in 1900: Hilbert.
- Formalization often obscures applications...
- ... but it shouldn't: formalization and algorithms are complementary.

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History of Numerical Analysis

Numerical Analysis has been applied:

- Numerical Analysis = Computers + Analysis + Large problems: von Neumann (1947).
- Techniques in Numerical Analysis are intrinsically algorithmic.
- The need for Numerical Analysis in applications is obvious.
- Numerical Analysis is often regarded as lacking rigour...
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New Trends in Didactics of Calculus

Calculus Reform:

- Rule of four: graphical, numerical, algebraical and verbal presentation.
- Incorporate technology.
- Concepts and techniques stem from practical problems.
- Active learning: encourage the students to explore and investigate.

New Trends in Didactics of Calculus

Body and Soul Project on Applied Mathematics:

- Computers should result in a paradigm shift in Mathematics.
- Applied Mathematics = Computational Mathematics.
- Constructive (computational) methods show that Mathematics are both understandable and useful.

Calculus and Numerical Analysis

Calculus and Numerical Analysis:

- Problem Oriented Learning: use all the knowledge relevant to the problem including (but not limited to) Calculus and Numerical Analysis.
- Active Learning: find *real* solutions to *real* problems, including a final numerical solution, physically interpretable.
- Computer usage is pervasive: Numerical Analysis should be transversal.

Calculus and Numerical Analysis

Calculus subject in an engineering degree:

- **Calculus of one variable:**

  Limits and continuity. Differentiation. Integration and methods of integration.

- **Infinite series:**

  Numerical series. Functional series. Power series. Taylor series. Fourier series.

- **Numerical methods:**

  Polynomial interpolation and approximation of functions. Numerical integration and differentiation. Roots.

- **Calculus of several variables:**

  Real and vector valued functions of several variables. Limits, continuity and differentiation.

- **Use of mathematical software:**

  Graphics. Curve fitting. Optimization.

Outline

Motivation

- Humans can recognise different objects from a given image made only by lines (like sketches).
- Human eye can make some king of edge and pattern recognition process to recognize forms and objects.
- Thus, edge detection techniques are required for a image to be processed by any automatic system.
- Applications:
  - Automatic patterns and form recognition by computer systems.
  - Tracking moving objects (for example in a security cam video).
  - Optical Character Recognition (OCR).
  - Images edition.

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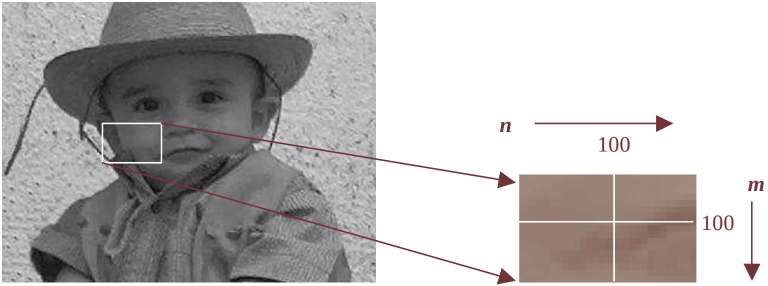
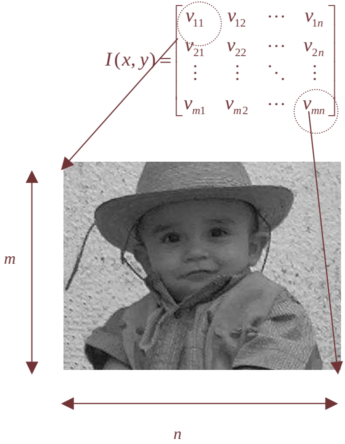
Digital images



Digital images



- Digital images consist of a number of pixels. They can be seen as a $m \times n$ matrix in which each element corresponds to a pixel.
- Information given by each pixel is coded by 8 bits, which corresponds to 256 grey levels, 0 corresponds to black and 255 to white.
- Let $I_{ij} = I(x_i, y_j)$ be the grey level corresponding to the pixel of coordinates (x_i, y_j), $i = 1, \ldots, m$, $j = 1, \ldots, n$ $(I_{ij} \in 0, 1, \ldots, 255)$.

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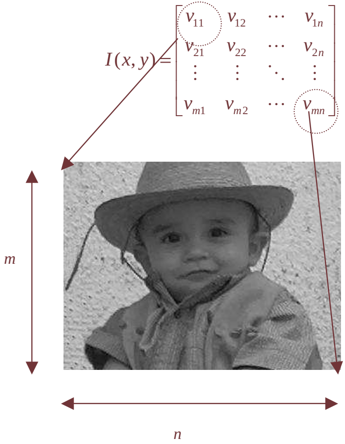

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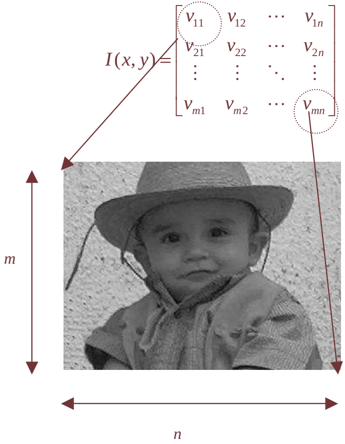

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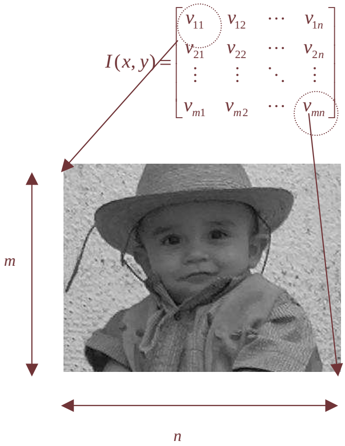

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$$I_R(m,n,1)=\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix} \quad I_G(m,n,2)=\begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{m1} & g_{m2} & \cdots & g_{mn} \end{bmatrix} \quad I_B(m,n,3)=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

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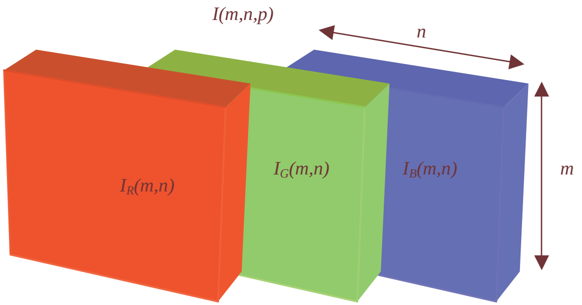

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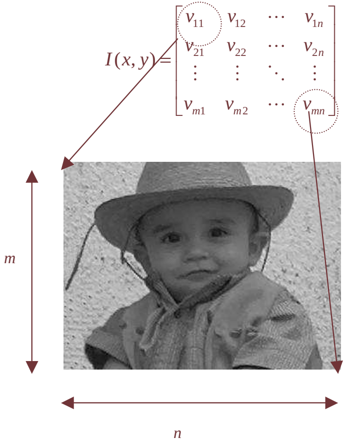

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- The gradient shows the variation of I. *Sharp* variations correspond to the existence of an edge.

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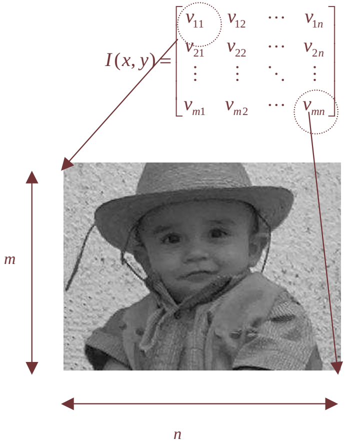

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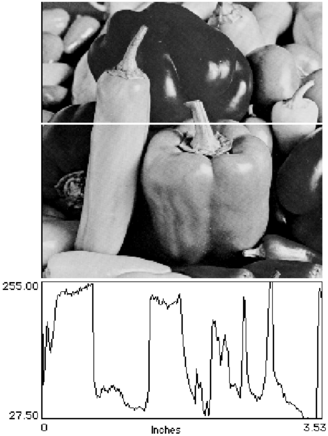
Definition

An edge may be defined as a sharp change in image brightness.

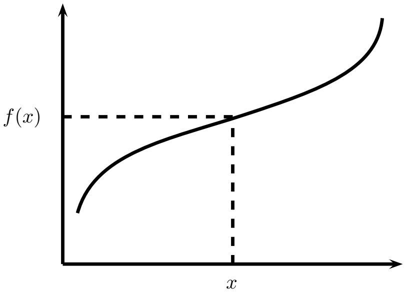


Outline

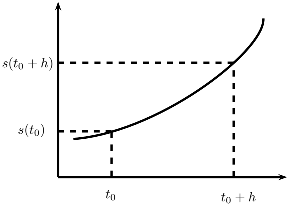
Functions of one variable

Definition

$$\begin{array}{rcl} f : D \subset \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longrightarrow & f(x) \end{array}$$



Derivatives



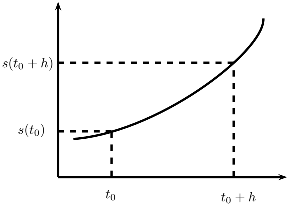
Average velocity

$$\frac{s(t_0 + h) - s(t_0)}{h}$$

Instant velocity

$$\lim_{h \to 0} \frac{s(t_0 + h) - s(t_0)}{h}$$

Derivatives



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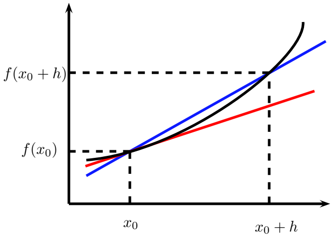
Derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Approximation

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}$$



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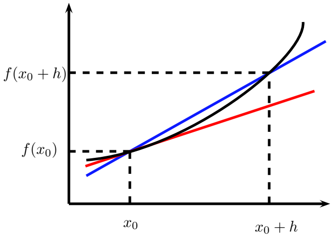

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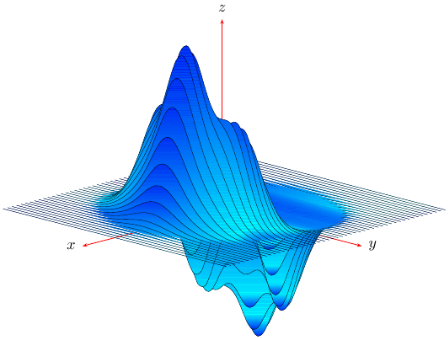

Partial derivatives and gradient

Partial derivatives

- $\dfrac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \dfrac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$
- $\dfrac{\partial f}{\partial y}(x_0, y_0) = \lim_{h \to 0} \dfrac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$

Gradient

$$\nabla f(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right)$$

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Vector valued functions of several variables

Definition

$$
\begin{array}{rcl}
f : D \subset \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\
(x, y) & \longrightarrow & (f_1(x, y), f_2(x, y), f_3(x, y))
\end{array}
$$

Outline

How to detect edges

- We use a numerical approximation of the gradient:

$$\nabla I(x_i, y_j) \approx \left(\frac{I(x_i + h, y_j) - I(x_i, y_j)}{h}, \frac{I(x_i, y_j + h) - I(x_i, y_j)}{h} \right).$$

- For the sake of simplicity we can assume that distance between pixels is equal to 1, that is, $h = 1$:

$$\nabla I(x_i, y_j) \approx (I_{i+1,j} - I_{i,j}, I_{i,j+1} - I_{i,j}).$$

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How to detect edges

- Once a numerical approximation of G_x and G_y is given, we shall assume existence of edges whenever the modulus of the gradient is greater than a given threshold level τ_0:

$$|G| = \sqrt{G_x^2 + G_y^2} \geq \tau_0 \quad \text{or} \quad |G| = |G_x| + |G_y| \geq \tau_0.$$

Masks

(G_x, G_y) may be represented by masks:

$(i-1, j-1)$	$(i-1, j)$	$(i-1, j+1)$
$(i, j-1)$	(i, j)	$(i, j+1)$
$(i+1, j-1)$	$(i+1, j)$	$(i+1, j+1)$

$$G_x = \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & 1 & -1 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \qquad G_y = \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & -1 & 0 \\ \hline \end{array}$$

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Masks

Other posibilities:

$$G_x = \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ \hline 0 & -1 & 0 \\ \hline \end{array} \quad G_y = \begin{array}{|c|c|c|}\hline -1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$$

$$G_x = \begin{array}{|c|c|c|}\hline 1 & 0 & -1 \\ \hline K & 0 & -K \\ \hline 1 & 0 & -1 \\ \hline \end{array} \quad G_y = \begin{array}{|c|c|c|}\hline -1 & -K & -1 \\ \hline 0 & 0 & 0 \\ \hline 1 & K & 1 \\ \hline \end{array}$$

Prewitt ($K = 1$), Sobel ($K = 2$), Frei-Chen ($K = \sqrt{2}$).

Remarks

- Edge detection is a more complex task than it seems.
- Usually pre-processing techniques are required such as noise reduction.
- The results will strongly depend on the mask used and on the threshold.
- There are other methods based on the second derivative (laplacian).

Conclusions

- Reverse the definition-axiom-theorem-example pattern: first propose the **problem**, then explore needed knowledge.
- The **problem** is *interesting* and *useful*: Maths are *interesting* and *useful*.
- Learn together Calculus and Numerical Analysis: they both are relevant to the **problem**.
- Mathematics do not lack rigour, but formalization is a *necessity* that arises during **problem** solution.
- Mathematics are not oversimplified: the **problem** often leans students towards *more* knowledge than standard syllabus.