# A CAS routine for obtaining eigenfunctions for Bryan's effect 

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#### Abstract

When a vibrating structure is rotated, the vibrating pattern of the structure rotates at a rate proportional to the rate of rotation of the structure. This effect, observed in 1890 by G.H. Bryan, is utilized in the vibratory gyroscopes that navigate space shuttles, submarines and commercial jetliners. In a recent article in the American Journal of Physics, expressions were derived for calculating Bryan's factor in terms of eigenfunctions that had not yet been determined. In this paper we numerically determine these eigenfunctions for the first few circumferential numbers as well as numerical values for Bryan's factor and the eigenfrequency of vibration. The numerical routine used here is more robust than "thin shell" theory but is easy enough for senior undergraduate students to understand and implement.


Keywords: Bryan's effect; Bryan's factor; Eigenfrequency; Eigenfunction; Boundary value problem, Thin shell theory.

## I. INTRODUCTION

It is intuitive to assume that the pitch and the pattern of a freely vibrating body such as a ring, singing wineglass, ringing bell, et cetera, does not change when the body starts a slow rotation. In mathematical terms we say that the "eigensystem of the vibration" does not change when slow rotation is introduced. However, this slow rotation does cause the vibration pattern to start to rotate within the body at a rate proportional to the rotation rate $\Omega$. This phenomenon was first observed by G.H. Bryan in $1890^{1}$. Nowadays this phenomenon is known as Bryan's effect and was discussedby Joubert, Shatalov and Fay ${ }^{2}$ for an annular body, where, in order to calculate Bryan's factor $\eta$ (the constant of proportionality mentioned above) it was assumed that $\Omega$ was small when compared to the lowest frequency of vibration. This lead

[^0]to neglecting terms of $O\left(\Omega^{2}\right)$ when an expression was derived for $\eta$ and the corresponding eigenvalue $\omega=2 \pi f$ (where $f$ is the frequency of vibration). The expressions for $\eta$ and $\omega$ were given in terms of unknown eigenfunctions in Ref. 2. In this paper we determine some of these eigenfunctions numerically for free vibration, as well as numerical values for the eigenfrequency of vibration as well as the associated Bryan's factor for an annular disk that is "quite thick" in the sense that the inner radius may be at least two thirds of the size of the outer radius. It is demonstrated that, for the free vibration of a slowly rotating "almost thin shell" cylinder, the routine appears to be more robust than the classical "thin shell" theory of Rayleigh ${ }^{3}$ and Bryan ${ }^{1}$. Indeed, as "thin shell" theory does for the fundamental vibration of the $m^{\text {th }}$ vibration mode ( $m=2, \cdots, 6$ ), the numerical routine produces four digit results for the eigenfrequency $f$ and the associated Bryan's factor $\eta$. On the other hand, at least three digit accuracy is obtained for $f$ and $\eta$ for each of the first overtone vibrations of these vibration modes. However, even though it pro-
duces the eigenfrequency to four digit accuracy for each of the second overtones, the numerical routine fails to produce results for the associated Bryan's factors.

It is important to be able to calculate Bryan's factor accurately because it is used to calibrate resonator gyroscopes that have many uses. Indeed, according to D.M Rozelle ${ }^{4}$, the resonator gyroscope "has been utilized in many applications over its developmental lifetime: aircraft navigation, strategic missile navigation, underground borehole navigation, communication satellite stabilization, precision pointing, and in deep space missions."

We have quantitatively checked the accuracy of our results as reported below. Indeed, neglecting terms of $O(\Omega)$ in order to simplify calculations, we have derived the eigensystem of the vibration quantitatively for any disk or annular disk (see Ref. 5) with clamped and/or free boundaries. In this article we show, among other results, that neglecting $O(\Omega)$ appears to be justified. Indeed, below we discuss an easy qualitative analysis in the form of numerical experiments that serves three purposes. Firstly, for a given mode of free vibration, it determines the fundamental and first overtone eigenfrequencies of vibration as well as the associated eigenfunctions of the vibration. Secondly, from these numerical calculations we deduce Bryan's factor $\eta$ for each of these frequencies. Thirdly, we can use this routine to show that including or neglecting slow inertial rotation $\Omega$ in the equations leads to "numerically identical" results. We use the words "numerically identical" to mean that results agree numerically to at least three and possibly four significant digits.

It is of interest to note that in the 1894 edition of his "Theory of Sound Volume I, §233", Rayleigh ${ }^{3}$ mentions, Bryan's effect. However, applications of Bryan's effect appear to have lain dormant for about 100 years and reappeared in the 1980's and 1990's (see for instance Scott ${ }^{6}$, Zhuravlev and Klimov ${ }^{7,8}$, and

Loper and Lynch ${ }^{9}$ ).

## II. A BOUNDARY VALUE PROBLEM

In Figure 1 of Ref. 2 the subscripts $i$ run from 1 to $N$. In this article we will consider a single annular ring (cylindrical disk) and so $N=1$. Consequently, using the notation of Ref. 2, for the single layer we set the inner radius $a_{0}=p$, outer radius $a_{1}=q$ and height $h_{1}=h$. The mass element with volume $d V$ under consideration at point $P$ is $\rho d V$ where $\rho$ is the density of the annular cylinder. The volume element $d V$ is given by Eq. B2 and sketched in Figure 3 of Ref. 2. Because $N=1$ we simplify notation and drop all subscripts. For instance we denote radial displacement by $u$ and tangential displacement by $v$. The situation is described in Fig. 1. In Eqs. (14)


FIG. 1: The polar coordinates $r$ and $\varphi$ of the position of rest $P$ of a vibrating particle in the annular disk.
and (15) of Ref. 2 we assumed that

$$
\begin{align*}
u & =U(r)[C(t) \cos m \varphi+S(t) \sin m \varphi]  \tag{1}\\
v & =V(r)[C(t) \sin m \varphi-S(t) \cos m \varphi] \tag{2}
\end{align*}
$$

where $U(r)$ and $V(r)$ were unknown eigenfunctions and $m$ is the circumferential wave number. In this paper we introduce an easy numerical routine for calculating these eigenfunctions.

Keep in mind that

$$
A \cos \omega t+B \sin \omega t=D \cos (\omega t+\theta)
$$

for arbitrary constants $A$ and $B$ and suitable constants $D$ and $\theta$. Then Problem 3 of Ref. 2 yields that

$$
\begin{align*}
C(t) & =D \cos \eta \Omega t \cos (\omega t+\theta)  \tag{3}\\
S(t) & =D \sin \eta \Omega t \cos (\omega t+\theta) \tag{4}
\end{align*}
$$

where $\eta$ is Bryan's factor. Using Eqs. (3) and (4) and double angle identities, we may now write Eqs. (1) and (2) as
$u(r, \varphi, t)=D U(r) \cos (m \varphi-\eta \Omega t) \cos (\omega t+\theta)$
$v(r, \varphi, t)=D V(r) \sin (m \varphi-\eta \Omega t) \cos (\omega t+\theta)$.
where $m$ is the circumferential wave number. The equations of motion for the mass element $\rho d V$ can be determined from Newton's second law of motion (per unit volume) by considering the components of stress in the radial and tangential directions, as, for instance, derived in Benham and Crawford ${ }^{10}$ :

$$
\begin{align*}
\rho \frac{\partial^{2} u}{\partial t^{2}} & =\frac{\partial \sigma_{r}}{\partial r}+\frac{1}{r} \frac{\partial \tau_{r \varphi}}{\partial \varphi}+\frac{\sigma_{r}-\sigma_{\varphi}}{r}  \tag{7}\\
\rho \frac{\partial^{2} v}{\partial t^{2}} & =\frac{\partial \tau_{r \varphi}}{\partial r}+\frac{1}{r} \frac{\partial \sigma_{\varphi}}{\partial \varphi}+\frac{2 \tau_{r \varphi}}{r}, \tag{8}
\end{align*}
$$

where $\sigma_{r}$ is radial stress, $\sigma_{\varphi}$ is tangential stress and $\tau_{r \varphi}$ is shear stress as described in the Appendices of Ref. 2.

Problem 1. Keeping in mind that we have only one layer and that we drop all subscripts, use Eqs. (9), (10), (11) and (12) of Ref. 2 in Eqs. (7) and (8) to obtain two coupled partial
differential equations (PDEs):

$$
\begin{array}{r}
\rho \frac{\partial^{2} u}{\partial t^{2}}=\frac{E}{2 r^{2}\left(1-\mu^{2}\right)}\left(\begin{array}{c}
2 r \frac{\partial u}{\partial r}+2 r^{2} \frac{\partial^{2} u}{\partial r^{2}}+ \\
(1+\mu) r \frac{\partial^{v}}{\partial r \partial \varphi}+ \\
(1-\mu) \frac{\partial^{2} u}{\partial \varphi^{2}}-2 u \\
-(3-\mu) \frac{\partial v}{\partial \varphi}
\end{array}\right) \\
\rho \frac{\partial^{2} v}{\partial t^{2}}=\frac{E}{2 r^{2}\left(1-\mu^{2}\right)}\left(\begin{array}{c}
(1+\mu) r \frac{\partial^{2} u}{\partial r \partial \varphi}+ \\
(1-\mu) r \frac{\partial v}{\partial r}+ \\
(1-\mu) r)^{2} \frac{\partial^{2} v}{\partial r^{2}}+ \\
(3-\mu) \frac{\partial u}{\partial \varphi}+ \\
2 \frac{\partial^{2} v}{\partial \varphi^{2}}-(1-\mu) v
\end{array}\right) \tag{10}
\end{array}
$$

where $E$ is young's modulus of elasticity and $\mu$ is Poisson's ratio for the annulus.

Recall that Ref. 2 demonstrated that Bryan's factor $\eta$ satisfies $|\eta| \leq 1$ and that $\omega$ was an eigenvalue of the vibrating system. This was achieved by assuming that the inertial rotation rate of the annulus $\Omega$ was significantly smaller than lowest eigenvalue of the system and then neglecting terms of $O\left(\Omega^{2}\right)$.

We substitute Eqs. (5) and (6) into (9) and (10) to obtain the following system of coupled differential equations (neglecting terms of $O\left(\Omega^{2}\right)$ ):

$$
\begin{align*}
& 0=2 r^{2} U^{\prime \prime}(r)+2 r U^{\prime}(r)+  \tag{11}\\
& \quad\left(a r^{2}-b\right) U(r)+c r V^{\prime}(r)-d V(r)  \tag{12}\\
& 0=(1-\mu) r^{2} V^{\prime \prime}(r)+(1-\mu) r V^{\prime}(r)+  \tag{13}\\
& \left(e r^{2}-f\right) V(r)-c r U^{\prime}(r)-d U(r) \tag{14}
\end{align*}
$$

where

$$
\begin{align*}
& a=\frac{2}{E}\left(1-\mu^{2}\right) \rho \times \\
& \binom{\omega^{2}+2 \eta \Omega \omega \times}{\tan (\omega t+\phi) \tan (m \varphi-\eta \Omega t)}  \tag{15}\\
& b=m^{2}(1-\mu)+2 \\
& c=m(1+\mu) \\
& d=m(3-\mu) \\
& e=\frac{2}{E}\left(1-\mu^{2}\right) \rho \times \\
&  \tag{16}\\
& \binom{\omega^{2}-2 \eta \Omega \omega \times}{\tan (\omega t+\phi) \cot (m \varphi-\eta \Omega t)}  \tag{17}\\
& f=2 m^{2}+1-\mu
\end{align*}
$$

Let us assume that values for $\omega$ and $\eta$ can be estimated. If we fix a value for $\phi$, choose a circumferential wave number $m$, a small inertial rotation rate $\Omega$ and a fixed point in time $t$, then all of the coefficients in Eqs. (17) are known. Consequently, Eqs. (11) and (13) constitute a system of linear ordinary differential equations (ODE). Notice that a form of Bessel's differential equation (see Ref. 11) is entwined in Eqs. (11) and (13). Pursuing this idea involves uncoupling the two PDEs (9) and (10) by introducing "potential functions" and then solving two wave equations that have solutions in terms of Bessel functions. This is a technically quite challenging and will not be attempted here. If we assume that the inner radius of the cylindrical disk is not too small when compared to the outer radius with say $\frac{2}{3} q \leq p<q$ and supply free boundary values, then the NDSolve routine of the commercial software programme Mathematica ${ }^{\circledR}$ solves the boundary value problem $(B V P)$ numerically to produce the eigenfunctions $U(r)$ and $V(r)$. In turn, $U(r)$ and $V(r)$ will be used to make a good approximation for the corresponding lowest eigenvalue $\omega$ and Bryan's factor $\eta$ for the chosen circumferential wave number $m$. By the words "good approximation" we mean numerical results that we trust to have at least four significant figures of accuracy.

Two naturally occurring boundary conditions exist for the annulus described in Figure (1), namely clamped boundaries and/or free boundaries. If a boundary is clamped, then no radial or tangential displacements can occur. Consequently a clamped inner boundary, for instance, would be represented by declaring that $U(p)=0, V(p)=0$. It is wellknown that a free boundary occurs (on the inner or outer edge) for the cylindrical disk in question when radial stress $\sigma_{r}=0$ and shear stress $\tau_{r \varphi}=0$. The next problem tells us how to represent free boundaries.

Problem 2. Keep in mind that we have only one layer and that we drop all subscripts. Now substitute Eqs. (11) and (12) into Eqs.
(9) and (10) of Ref. 2 and use Eqs. (5) and (6) to show that

$$
\begin{equation*}
\sigma_{r}=0 \Longrightarrow \mu U(r)+m \mu V(r)+r U^{\prime}(r)=0 \tag{18}
\end{equation*}
$$

$\tau_{r \varphi}=0 \Longrightarrow m U(r)+V(r)-r V^{\prime}(r)=0$.

Our numerical routine appears to work only for the cylindrical disks described above with "free boundaries" and so we limit ourselves here to such problems. Analytical methods that solve "free and/or clamped boundary value problems" for disks and annular disks are dealt with in Ref. 5.

## III. EXAMPLE

Consider an aluminium annular disk rotating at $\Omega=\pi$ rad. $\mathrm{s}^{-1}$ with inner radius $p=0.14 \mathrm{~m}$, outer radius $q=0.15 \mathrm{~m}$, height $h=0.01 \mathrm{~m}$, density $\rho=2700 \mathrm{~kg} . \mathrm{m}^{-3}$, Poisson's ratio $\mu=\frac{1}{3}$, Young's modulus $E=7 \times$ $10^{10} \mathrm{~Pa}$, circumferential wave number $m=2$, phase angle $\phi=1 \mathrm{rad}$ and time $t=1 \mathrm{~s}$. Use "first guess" values $\omega=0 \mathrm{rad} . \mathrm{s}^{-1}$ and $\eta=1$ for the BVP given by Eqs. (11), (13), (18) and (19). Now use the "NDSolve" routine of Mathematica ${ }^{\circledR} 7$ with default settings to numerically solve this BVP. In order to produce a nonzero solution using the "shooting method", include the command: Method $\rightarrow$ \{"Shooting", "StartingInitialConditions" $\rightarrow$ $\left\{U[p]==0, U^{\prime}[p]==1, V[p]==0, V^{\prime}[p]==\right.$ $1\}\}$. The reader may observe that any, not all zero, "StartingInitialConditions" also produce a suitable nonzero solution. Such a Mathematica ${ }^{\circledR}$ solution is in the form of "Interpolation functions" that are stored in the computer's memory. This is convenient because it is now an almost effortless exercise to recall them and numerically calculate "better" values for $\omega$ and $\eta$ using the "NIntegrate" routine of Mathematica ${ }^{\circledR}$. We achieve this by numerically calculating the formulae

$$
\begin{equation*}
\omega=\sqrt{\frac{I_{2}}{I_{0}}} \tag{20}
\end{equation*}
$$

and

$$
\begin{equation*}
\eta=\frac{I_{1}}{I_{0}} \tag{21}
\end{equation*}
$$

where

$$
\begin{gather*}
I_{0}=\frac{1}{2} h \rho \int_{p}^{q}\left(U^{2}+V^{2}\right) r d r,  \tag{22}\\
I_{1}=h \rho \int_{p}^{q} U V r d r \tag{23}
\end{gather*}
$$

and

$$
\begin{align*}
I_{2}= & \frac{1}{2} \frac{E h}{1-\mu^{2}} \int_{p}^{q}\left\{\left(U^{\prime}\right)^{2}+2 \mu U^{\prime} \frac{U+m V}{r}+\right. \\
& \left(\frac{U_{i}+m V}{r}\right)^{2}+  \tag{24}\\
& \left.\frac{1-\mu}{2}\left(V^{\prime}-\frac{m U+V}{r}\right)^{2}\right\} r d r . \tag{25}
\end{align*}
$$

These formulae are to be found in Ref. 2 Eqs. (26), (27), (18), (19) and (20) respectively. The "NDSolve" routine is now recalculated using the updated values of $\omega$ and $\eta$ determined by Eqs. (20) and (21). After about five iterations, the process appears to converge to results that have at least four digits of agreement between successive iterations, namely (for the $m=2$ mode of vibration) the negative Bryan's factor

$$
\begin{equation*}
\eta \approx-0.7990 \tag{26}
\end{equation*}
$$

and angular rate of vibration

$$
\begin{equation*}
\omega \approx 1872 \mathrm{rad} . \mathrm{s}^{-1} \tag{27}
\end{equation*}
$$

The fundamental frequency of vibration for the $m=2$ mode of vibration is thus

$$
\begin{equation*}
f \approx 297.9 \mathrm{~Hz} \tag{28}
\end{equation*}
$$

Plots of $U(r)$ and $V(r)$ (scaled by their respective maximum modulus that was also calculated by Mathematica ${ }^{\circledR}$ ) appear in Figs. (2) and (3). Note that, depending on
how many iterations are made and/or what version of Mathematica ${ }^{\circledR}$ is used, it might not necessarily return unique functions $U(r)$ or $V(r)$. However, the numerical values of $\omega$ and $\eta$ should be unique up to four significant figures for a given mode of vibration $m$.


FIG. 2: Eigenfunction $U(r)$ scaled by its maximum modulus.


FIG. 3: Eigenfunction $V(r)$ scaled by its maximum modulus.

## IV. CHECKING RESULTS

In 1877 Rayleigh ${ }^{3}$ derived a formula for the fundamental frequency of vibration for the $m^{\text {th }}$ mode of the vibration of a "thin shell" cylinder with free boundaries, assuming the complete inextensibility of the circumference (Eq. (9), § 233). The disk in Example (III) approximates a "thin shell" cylinder, that is $q-p \ll 2 R$, where $R=\frac{p+q}{2}$ is the middle-line radius of curvature. The surface area of the $r, z-$ face is $S=h(q-p)$ and has a
second moment of inertia $J$ about the middle line given by $J=\frac{h(q-p)^{3}}{12}$ (see Ref. 12). Zhuravlev and Klimov ${ }^{7}$ modified Rayleigh's Eq. (9), § 233, to include (not necessarily small) rotation of the cylinder $\Omega$ as follows:

$$
\begin{equation*}
f=\frac{m\left(m^{2}-1\right)}{2 \pi\left(m^{2}+1\right)} \sqrt{\frac{\left(m^{2}+1\right) E J}{\rho S R^{4}}+\Omega^{2}} . \tag{29}
\end{equation*}
$$

Substituting the values given in Example (III) yields

$$
\begin{equation*}
f \approx 299 \mathrm{~Hz} \tag{30}
\end{equation*}
$$

rounded to three significant digits. This agrees reasonably well with Eq. (28). Furthermore, if $\dot{\Theta}_{m}$ is the angular rate of rotation of the nodes when observed from within the rotating system, then an observer in fixed space would see that the nodes rotate at a rate of $\dot{\Theta}_{m}+\Omega$. In 1890, using the same assumptions as Rayleigh ${ }^{3}$ for a thin shell, Bryan ${ }^{1}$ calculated that

$$
\begin{equation*}
\dot{\Theta}_{m}+\Omega=\frac{m^{2}-1}{m^{2}+1} \Omega \tag{31}
\end{equation*}
$$

and so

$$
\begin{equation*}
\dot{\Theta}_{m}=-\frac{2}{m^{2}+1} \Omega \tag{32}
\end{equation*}
$$

The calculations made for a spherical body by Shatalov, Joubert and Coetzee ${ }^{13}$ may be readily adapted for the annulus described by Figure (1). For an ideal annulus (with or without the inclusion of isotropic viscous damping into the equations of motion), Eq. (51) of Ref. 13 yields

$$
\begin{equation*}
\dot{\Theta}_{m}=\frac{\eta}{m} \Omega . \tag{33}
\end{equation*}
$$

Consequently, for a "thin shell" cylinder

$$
\begin{equation*}
\eta=-\frac{2 m}{m^{2}+1} \tag{34}
\end{equation*}
$$

Notice that "thin shell" theory predicts that Bryan's factor $\eta$ is negative and has the same size for the fundamental vibration of the $m^{t h}$ mode of vibration for all thin shells, no matter what the radius of curvature is. For instance, with $m=2$ we obtain

$$
\begin{equation*}
\eta=-0.8 \tag{35}
\end{equation*}
$$

agreeing reasonably well with Eq. (26).
Further evidence of the validity of the method used in Example (III) is given by the commercial software programme COMSOL Multiphysics $4.0^{\circledR}$ set to "Extremely fine" predefined mesh element size. Indeed, one of the frequencies of "free" vibration of the disk described by Example (III) is the same (up to four significant figures) as that given by Eq. (28) (see Fig. (4)). Notice that Fig. (4) indicates that the vibration pattern has four nodes (the grey parts on the deformed ring), agreeing with the $m=2$ mode of vibration (the number of nodes in the fundamental $m^{\text {th }}$ mode of vibration is $2 m$ ).

Eigenfrequency=297.87846


FIG. 4: The COMSOL Multiphysics 4.0 graph of the annulus for the $m=2$ mode of vibration, showing four nodes and the lowest eigenfrequency of vibration $f=297.878513 \mathrm{~Hz}$.

Problem 3. Use the values and process described in Example (III) but make variations as follows:

1. Use "StartingInitialConditions" $\rightarrow\left\{U[p] \quad==200, U^{\prime}[p]==\right.$ $\left.-150, V[p]==1 / 2, V^{\prime}[p]==500\right\}$ to illustrate the statement above that "any, not all zero, "StartingInitialConditions"" yield a suitable nonzero solution.
2. Although the numerical routine might not appear to return unique functions
$U(r)$ and $V(r)$, show (by adapting Example (III)), that the results obtained above for $\eta$ and $\omega$ are numerically identical up to four significant digits if we choose $\Omega=0 \mathrm{rad} . \mathrm{s}^{-1}$ or $\Omega=$ $10 \pi$ rad. $\mathrm{s}^{-1}$. This illustrates the statement that the eigensystem discussed in the introduction is not effected by a "small" inertial rotation rate $\Omega$.
3. For $p=9 \mathrm{~cm}$ and $q=10 \mathrm{~cm}$, let $m=2$ and show that the fundamental frequency appears to be

$$
\begin{equation*}
f \approx 0.6919 \mathrm{kHz} \approx 0.7 \mathrm{kHz} \tag{36}
\end{equation*}
$$

with

$$
\begin{equation*}
\eta \approx-0.7977 \approx-0.8 \tag{37}
\end{equation*}
$$

Verify that the "thin shell" theory yields

$$
\begin{equation*}
f \approx 0.69556 \mathrm{kHz} \approx 0.7 \mathrm{kHz} \tag{38}
\end{equation*}
$$

and

$$
\begin{equation*}
\eta \approx-0.8 \tag{39}
\end{equation*}
$$

4. For $p=14 \mathrm{~cm}$ and $q=15 \mathrm{~cm}$ let:
(a) $m=3(m=6$ respectively) and show that the fundamental frequency appears to be
$f \approx 0.8393 \mathrm{kHz} \quad(f \approx 3.740 \mathrm{kHz})$
and the corresponding Bryan's factor is

$$
\begin{equation*}
\eta \approx-0.5977 \quad(\eta \approx-0.3188) \tag{41}
\end{equation*}
$$

Verify that "thin shell" theory Eqs. (29) and (34) agree reasonably well with these results.
(b) $m=2$ and assume "guess values"
i. $\omega=10^{5} \mathrm{~s}^{-1}$ and $\eta=1$. Show that we obtain another vibration frequency (this is actually the first overtone vibration frequency of the $m=2$ mode of vibration)
$f \approx 12.48 \mathrm{kHz}$
and the corresponding Bryan's factor is positive with

$$
\begin{equation*}
\eta \approx 0.796 \tag{43}
\end{equation*}
$$

Notice that one cannot check the validity of this method using Eqs. (29) and (34) because these formulae are only valid for the fundamental frequency of vibration of a thin shell cylinder. However, we have checked these results using an analytical method (see Ref. 5) and they are accurate to at least three significant digits.
ii. $\omega=10^{6} \mathrm{~s}^{-1}$ and $\eta=1$. Show that we obtain another vibration frequency (this is actually the second overtone vibration frequency of the $m=$ 2 mode of vibration)
$f \approx 156.5 \mathrm{kHz}$.
Unfortunately the iteration process does not appear to converge for the calculation of Bryan's factor. Indeed, there appears to be no agreement in the third decimal place, with values ranging from $0.002 \ldots$ to $0.005 \cdots$. This illustrates that there is little reliability in the result for $\eta$. We have checked this result against an analytical method (see Ref. 5) and found that the frequency shown here is accurate to four significant digits but the value for Bryan's factor is

$$
\begin{equation*}
\eta \approx 0.0006087 \approx 0.001 \tag{45}
\end{equation*}
$$

(c) $m=4$ and assume "guess values":
i. $\omega=0 \mathrm{~s}^{-1}$ and $\eta=1$. This yields the fundamental vibration frequency
$f \approx 1.601 \mathrm{kHz}$
and the corresponding Bryan's factor is negative with

$$
\begin{equation*}
\eta \approx-0.467 \tag{47}
\end{equation*}
$$

ii. $\omega=10^{5} \mathrm{~s}^{-1}$ and $\eta=1$ ". This yields the first overtone vibration frequency
$f \approx 23.00 \mathrm{kHz}$
and the corresponding Bryan's factor is positive with

$$
\begin{equation*}
\eta \approx 0.461 \tag{49}
\end{equation*}
$$

iii. $\omega=10^{6} \mathrm{~s}^{-1}$ and $\eta=1$ ". This yields the second overtone vibration frequency
$f \approx 158.0 \mathrm{kHz}$.
For the calculation of Bryan's factor, even though the iteration process appears to convergent to two significant digits after five iterations, if the process is continued, then the sixth iteration shows that the calculation changes from $0.0049 \cdots$ to $0.01 \cdots$, a change of more than $100 \%$ (called an order of magnitude change)! Continuing with the iteration process, after nine iterations the process appears to have converged again to a value close to 0.0049 . This observation should warn us that there is something amiss. On the other hand, the value of the frequency has already
converged to four significant digits after five (or even many more) iterations, without any change in the significant digits. We have checked this result against an analytical method (see Ref. 5) and found that the frequency is accurate to four significant digits but the true value for Bryan's factor is

$$
\begin{equation*}
\eta \approx 0.00139007 \tag{51}
\end{equation*}
$$

## V. CONCLUSION

By solving a boundary value problem we determined the eigensystem and corresponding Bryan's factor for the fundamental and first overtone vibrations of a "quite thick" cylindrical shell (by "quite thick" we mean that the inner radius is approximately two thirds of the size of the outer radius) with free boundaries. This was done for low circumferential wave numbers ( $m=1, \cdots, 6$ ). For second overtone vibrations (in our example this means that frequencies are more than 100 kHz ) the method fails to calculate Bryan's factor, although it appears to calculate all pertinent vibration frequencies correctly to four significant figures of accuracy. The method used is easy enough for senior undergraduate students to master, enabling them to understand how vibratory gyroscopes (that are used for instance in the navigation of deep space probes) are calibrated. This numerical method appears to be more robust than the "thin shell" methods that have been utilised in the past.

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expressed in this material are those of the authors and therefore the TUT, the CSIR and the NRF do not accept any liability in regard thereto.
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