## 1. FIRST ORDER LINEAR HOMOGENOUS DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

In this section we investigate problems that are modeled by first-order linear homogenous difference equations with constant coefficients. That is equations in the form:

$$
\begin{equation*}
y_{n+1}=a y_{n} \tag{1}
\end{equation*}
$$

Where $a$ is a constant coefficient. We will iterate equation (1) with an initial condition to find a numerical solution and also we will derive an analytical solution of equation (1).

## DRUGS

## ACTIVITY 1

Assume that the kidney remove $20 \%$ of a drug from the blood every four hours and assume that the initial dose of the drug is 200 milligrams ( mg ). Let $y_{n}$ denote the amount of the drug in the blood after $n$ four-hour periods, and $y_{0}$ denote the initial amount of the drug in the blood.
(A) Find a difference equation that represents this situation.
(B) Find the amount of drug in the blood after 12 hours (i.e. 3 four-hour periods).
(C) Iterate the difference equation obtained in (A) with the initial condition to find the numerical solution $\left[n, y_{n}\right], n=0,1,2, \ldots, 20$. Graph it and describe the graph.
(D) Find an analytical solution of the obtained difference equation. Use this solution to find the amount of drug in the blood after one day.
(E) When will the amount of the drug reach 1 mg ?

## SOLUTION

(A) The amount of the drug in the blood after $(n+1)$ four-hour periods, $y_{n+1}$, equals the amount of the drug after $n$ four-hour periods, $y_{n}$, minus $20 \%$ of $y_{n}$. This situation is modeled by the difference equation (2) and the initial condition (3):

$$
\begin{align*}
& y_{n+1}=y_{n}-0.20 y_{n}=0.80 y_{n}  \tag{2}\\
& y_{0}=200 \tag{3}
\end{align*}
$$

(B)
(C) You may use DERIVE's function ITERATES to find the numerical solution $\left[n, y_{n}\right.$ ], $n=0,1,2, \ldots, 20$. Note that $0.8 y$ updates $y, y_{0}=200, k=20$. Do the following:

Author ITERATES ([n + 1, 0.8y], [n, y], [0, 200], 20)
Approximate
Plot
(D) The analytical solution of (2) is

$$
\begin{equation*}
y_{n}= \tag{4}
\end{equation*}
$$

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The amount of the drug in the blood after one day $=\mathrm{mg}$.
(E) The amount of drug in the blood reaches 1 mg after approximately . four-hour periods, i.e. after $\qquad$ hours or $\qquad$ days.

## ANALYTICAL SOLUTION

Students derive the analytical solution of the difference equation (1) in the form:

$$
\begin{equation*}
y_{n}=a^{n} y_{0} \tag{5}
\end{equation*}
$$

Analytically they can check that equation (5) is a solution of equation (1) by showing that $y_{n}$ in (5) satisfies equation (1).

## DIFFERENCE EQUATIONS AND DERIVE-SECOND PASS

In section 1, we discussed how a numerical solution of a first-order difference equation might be generated with DERIVE. Now we will see how to use DERIVE to find an analytical solution of a first-order linear difference equation in the form:

$$
\begin{equation*}
y_{n+1}=a_{n} y_{n}+b \tag{6}
\end{equation*}
$$

The utility file Recureqn contains the function
LIN1_DIFFERENCE $\left(a_{n}, b_{n}, n, n_{0}, y_{0}\right)$
that simplifies to a specific solution of (6) with the initial conditions $y_{0}$, where $n_{0}$ is the initial value of $n$. You need to load this utility file into your Algebra windows in order to be able to use the predefined functions in that file. To load this file:

## Click on File $>$ Load $-\underline{\text { Utility }}$ •Recuregn.MTH $-\underline{O p e n}$

Let us see how to find the analytical solution of the difference equation (2): $y_{n+1}=0.8 y$, with the initial condition (3): $y_{0}=200$. The parameter function LIN1_DIFFERENCE are:
$a_{n}=0.8, b_{n}=0, n=n, n_{0}=0, y_{0}=200$. Do the following:
Author LIN1_DIFFERENCE ( $0.8,0, n, 0,200$ )
Simplify (by clicking on = icon)
The output is the expression:
To evaluate, for example $y_{20}$ substitute 20 for $n$ in the above expression. Do the following:
Click on Simplify Substitute for $\underline{\text { Variables }}$
Type in 20 in the window Substitution
Click OK
Approximate (by clicking $\approx$ icon)
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The output is:

## 2. POPULATION DYNAMICS-FIRST PASS

## ACTIVITY 2

Assume that the population of flies each summer is related to the population of flies from the previous summer. Assume that the initial population is 100,000 flies.
For each of the situations (i), (ii), and (iii):
(i) The flies increase by $10 \%$ every summer.
(ii) The flies decrease by $8 \%$ every summer.
(iii) The population of flies is the same each summer.
(A) Represent the situation by a difference equation and an initial condition.
(B) Find an analytical solution and use it to find the population of flies after 10 summers.
(C) Find a numerical solution $n=0,1,2, \ldots, 20$ and graph it. Describe the graph.

## SOLUTION

Let $P_{n}$ be the population of flies in thousands in the $n^{\text {th }}$ summer.

Situation (i)
(A) This situation is modeled by the difference equation:

$$
P_{n+1}=
$$

and the initial condition $P_{0}=$
(B) The analytical solution of the difference equation is

$$
P_{n}=\quad \text { and } P_{10}=
$$

(C)

Situation (ii)
(A)
(B)
(C)

Situation (iii)
(A)
(B)
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## (C)

## CARBON DATING

The following principle is used by archeologists to estimate the age of once living archeological finds such as bone or wood. There are two isotopes of carbon: radioactive carbon-14 and nonradioactive carbon-12. Carbon-14 and carbon-12 are absorbed in small amounts by all living tissues (animal and plant). As long as an animal or plant is alive, the ratio of carbon-14 to carbon-12 is a fixed constant and is the same as in the atmosphere. When an organism dies no new carbon-14 is absorbed by the organism, and the existing carbon-14 slowly decays. The halflife of carbon-14 is 5,730 years. The ratio of carbon-14 to carbon-12 in a fossil is measured and can be used to estimate the age of the fossil, (i.e. the date when the organism died). Note that the ratio of carbon-14 to carbon-12 in the atmosphere is constant over the millennia. This principle was developed in 1946 by Willard Libby who received a Nobel Prize in chemistry.

## ACTIVITY 3

The laboratory testing revealed that $30 \%$ of carbon-14 is missing from a bone fragment. Knowing that the half-life of carbon-14 is 5,730 years, how old is the bone?

## SOLUTION

Assume that the carbon-14 decays at a constant rate, $d$, per year. Let $y_{n}$ be the amount of carbon-14 in the bone after $n$ years, where $n$ is measured from the death year of the animal. We have

$$
y_{n+1}=y_{n}-d y_{n}=(1-d) y_{n}
$$

Letting $1-d=c$, we get $y_{n+1}=c y_{n}$. Use the analytical solution of this difference equation and the half-life of carbon-14 to determine the value of the constant $c$ :

$$
c=
$$

Use the given information that the amount of carbon-14 in the bone now is $y_{n}=0.70 y_{0}$ to determine the age of the bone

## 2. FIRST ORDER LINEAR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

In this section we investigate some mathematical models that are represented by first-order linear nonhomogeneous difference equations in the form:

$$
\begin{equation*}
y_{n+1}=a y_{n}+b \tag{1}
\end{equation*}
$$

where $a$ and $b$ are constants

## DRUGS

## ACTIVITY 1(i)

Suppose that the kidney remove 20\% of a drug in the blood every four hours. Assume that a patient takes an initial dose of a drug followed by a dose of 20 mg of the same drug every four hours. Assume that the initial dose of the drug is 140 mg . Let $y_{n}$ be the amount of drug in the blood at the end of $n$ four-hour periods.
(A) Find a difference equation that represents the situation. Use the difference equation to determine the amount of drug in the blood after 20 hours.
(B) Find a numerical solution $\left[n, y_{n}\right], n=0,1,2, \ldots, 25$. Graph this solution and describe the graph.
(C) Algebraically show that the analytical solution of the difference equation is

$$
y_{n}=\left(y_{0}-100\right)(0.8)^{n}+100
$$

Use DERIVE to find the analytical solution of the difference equation. What is the amount of drug in the blood after two days?

## SOLUTION

(A) This situation is modeled by the difference equation:
and the initial condition:
The amount of the drug in the blood after 20 hours is $\qquad$ mg.
(B)
(C) To find an analytical solution we have:

$$
\begin{aligned}
& y_{1}=0.8 y_{0}+20 \\
& y_{2}=0.8 y_{1}+20=0.8\left(0.8 y_{0}+20\right)+20=(0.8)^{2} y_{0}+[20(0.8)+20] \\
& =(0.8)^{2} y_{0}+[20+20(0.8)] \\
& y_{3}=0.8 y_{2}+20= \\
& y_{n}=
\end{aligned}
$$

## ACTIVITY 1(ii)

This situation is the same as in Example 1(i), but the initial dose of the drug is 70 mg .

## SOLUTION

(A)
(B)
(C)

## ACTIVITY 1(iii)

This situation is the same as in Activity 1(i), but with the initial dose of the drug is 100 mg .

## SOLUTION

(A)
(B)
(C)

## ANALYTICAL SOLUTION OF A FIRST-ORDER LINEAR DIFFERENCE EQUATION WITH CONSTANT COEFFICIENT

Consider the first-order linear difference equation (1). We have

$$
\begin{aligned}
& y_{1}=a y_{0}+b \\
& y_{2}=a y_{1}+b=a\left(a y_{0}+b\right)+b=a^{2} y_{0}+a b+b \\
& y_{3}=a y_{2}+b=a\left(a^{2} y_{0}+a b+b\right)+b=a^{3} y_{0}+a^{2} b+a b+b
\end{aligned}
$$

From this pattern we conclude that

$$
y_{n}=a^{n} y_{0}+a^{n-1} b+a^{n-2} b+\ldots+b=a^{n} y_{0}+b\left(a^{n-1}+a^{n-2}+\ldots+1\right)
$$

The analytical solution of the difference equation $y_{n+1}=a y_{n}+b$, where $a$ and $b$ are constants is

$$
y_{n}= \begin{cases}\left(y_{0}+b /(a-1)\right) a^{n}-b /(a-1) & a \neq 1  \tag{3}\\ y_{0}+b n & a=1\end{cases}
$$

## CONSTANT SOLUTIONS AND EQUILIBRIUM VALUES

In Activity 1(iii), we realized that if $y_{0}=100$, then $y_{1}=100, y_{2}=100, \ldots$ In other words

$$
y_{n}=100 \text { for } n=1,2, \ldots
$$

which means that the values of the solution do not change with time and remain constant at 100 . This solution is called a constant or steady-state solution. The constant value 100 is called an
equilibrium value of the difference equation. If a solution of a difference equation reaches the equilibrium value over a period of time it remains constant at the equilibrium value as in Activities 1(i) and 1(ii). Note that a difference equation, with initial values greater than or less than the equilibrium value, approach the equilibrium value, then the equilibrium value is called stable or attractor otherwise is called unstable or repeller. Let $E$ be the equilibrium value of the difference equation (1). To find $E$ put $y_{n+1}=y_{n}=E$ in (1) and solve for $E$. We have

$$
\begin{align*}
& E=a E+b \\
& E=b /(1-a) \tag{4}
\end{align*}
$$

## POPULATION DYNAMICS-SECOND PASS

## ACTIVITY 2

Assume that the annual growth rate of deer in Luzern County is $4 \%$ and the state restricts hunting to 8,000 deer every year. Let $P_{0}$ be the deer population in thousand in 1990 and $P_{n}$ be the deer population after $n$ years from 1990 .
(A) Model this situation by a difference equation. Find the analytical solution and an equilibrium value.
(B) For each of the following populations, generate the numerical solution $\left[n, P_{n}\right]$, $n=0,1,2, \ldots, 20$. Graph it and describe the graph.
a. 200,000
b. 250,000
c. 150,000
(C) Determine whether the equilibrium value is stable or unstable.

## SOLUTION

(A) This situation is modeled by the difference equation:

The analytical solution of this difference equation is:
The equilibrium value $E$ of this difference equation is $E$ :
(B) Describe the graph:
a.
b.
c.
(C)

