# Experiment with geometric loci 

Wolfgang Moldenhauer<br>Wolfgang.Moldenhauer@thillm.de<br>Thuringian Institut of Inservice Teacher Training, Curriculum Development and Media (ThILLM), Bad Berka, Germany

## Keywords:

Curriculum, Computer Algebra System, School improvement, Secondary school, Self regulated learning, Problem solving


#### Abstract

: Due to the availability of dynamic geometry software (DGS) and computer algebra systems (CAS) the discussion of geometric loci in school should be viewed from a different perspective. This could potentially stimulate the school improvement in the aspects "linking geometry, analysis, algebra", "experimental work" and "learning by discovery". The study of geometric loci can be an enrichment in the course of talented student programs and thereby improve distinction. An assessment of the difficulties to solve the respective question is not always possible since already for easy looking problems it occasionally turns out that e. g., finding an algebraic description of the geometric locus is difficult. On the contrary, there are also examples that are solvable with elementary school methods. The task for the lecturer is to provide his student with a competency gain. The given examples try to enlighten this problematic topic and simultaneously present some geometric loci from triangular geometry.


## Introduction and background

The discussion of geometric loci in school is not new and there are a lot of standard examples in the normal Thuringian curriculum. However, those examples mostly cover only isolated points of geometric theory and it will be highly beneficial to use more and different examples that allow students to link different fields of mathematics and learn by discovery. This paper tries to give such examples that show how dynamic geometry software (DGS) and computer algebra systems (CAS) can be used in school to link various topics and thus be used in school. The authors hope to provide a useful set of examples in the following that will support teachers in their day to day work.
We argue that with the availability of DGS and CAS the study of geometric loci in school curriculum should be reviewed as an enriching opportunity that will strengthen geometric understanding and capabilities of students. The authors see a big chance to stimulate the mathematical education with the study of geometric loci in at least in the following aspects:

- Linking geometry, analysis and algebra,
- Increase of experimental work, e. g. in projects,
- Fostering learning by discovery

The background against which we developed the following examples to study geometric loci is a change of educational policies in one of Germanies states: Thuringia.

Germany is divided into 16 states with different educational authorities and of course more than 16 different curriculums and final examinations. Depending on each states' educational policies sometimes teachers design the final exam for their own students. In other states there is a centralized exam that all students have to pass in order to graduate from highschool. In that respect a big variety of curricula exists in Germany and teachers have a varying degree of freedom to shape the content of their classes depending on the state educational policies.
Thuringia is one of Germanys' states which recently has gained additional potential in reshaping curriculums. An educational reform was rolled out in the last years in Thuringia with the goal of introducing dynamic geometry software (DGS) and computer algebra systems (CAS) to schools. In 2014 the usage of DGS and CAS will be mandatory in the final (central) exams for all students in Thuringia and thus the reform is a significant step towards introducing new teaching technology and methods and developing a new math curriculum.

## Suggested examples for the study of geometric loci

For all our suggested examples we will use a standard triangle. This triangle will serve as a reference considering all following examples. This is not a limitation because it is possible to map our standard triangle ABC with the vertices $\mathrm{A}(0 ; 0), \mathrm{B}(1 ; 0)$ and $\mathrm{C}(\mathrm{t} ; 1)$ with $\mathrm{t} \in \mathrm{R}$ to an arbitrary triangle XYZ with a one to one linear transformation.. In the following examples only $t$ is variable, which means that $C$ moves on a straight line (parallel to the $x$-axis). As said before for the following examples we only use this standard triangle ABC.
Our examples focus on the special lines in a triangle and our interest is concentrated only on altitudes $\left(h_{a}, h_{b}, h_{c}\right)$, angle bisections $\left(\omega_{\alpha}, \omega_{\beta}, \omega_{\gamma}\right)$, medians ( $\left.\mathrm{s}_{\mathrm{a}}, \mathrm{s}_{\mathrm{b}}, \mathrm{s}_{\mathrm{c}}\right)$, perpendicular bisectors of the sides $\left(\mathrm{m}_{\mathrm{a}}, \mathrm{m}_{\mathrm{b}}, \mathrm{m}_{\mathrm{c}}\right)$ and mixed variants of them. In the following we will look for intersection points and geometric loci of those special lines with variations in the parameter t . Our examples consider the heart curve, a mixed variant, the intersection point of the altitudes and the intersection point of the angle bisectors. Further examples can be found in [1], [2] an [3]. Of course it is possible to consider further variations of this problem set or even further to develop different settings by considering other lines in the triangle. To solve these types of geometric problems DGS as well as CAS are well suited.

## Example 1: The heart curve

Given is the standard triangle ABC as defined previously. Let be S the intersection point of the medians. Now point C of the triangle is moved on the heart curve (see Fig 1).
What is the geometrical locus of point $S$ caused by the motion of point $C$ ?
Answer:
It is a little heart. More precisely for each point C on the heart curve and its corresponding point S the following equation $\overline{\mathrm{CS}}: \overline{\mathrm{SM}}=2: 1$ is fulfilled. And by similarity we get that the given big heart curve is three times longer than the small one.
The mathematical explanation for this result is, that the intersection point $S$ of the medians $s_{a}$, $\mathrm{s}_{\mathrm{b}}, \mathrm{s}_{\mathrm{c}}$ divide the median proportional to $2: 1$.

Example 2: A mixed problem ( $\mathrm{S}=\mathrm{h}_{\mathrm{c}} \cap \mathrm{m}_{\mathrm{a}}$ )
How does S move when C is moved with its parameter t ?

Last weekend I worked together with students. One of the students presented the following solution: $\overline{\mathrm{CS}}=\overline{\mathrm{SB}}$. That means, that the point S has the same distance from a fixed point B and from a straight line. But this is the definition of a parabola. Therefore the geometrical locus of S is a parabola. The point B is the focus of this parabola.
A simple calculation with $h_{c}: x=t$ and $m_{a}: y=(1-t)\left(x-\frac{t+1}{2}\right)+\frac{1}{2}$ gives the equation $y=\frac{1}{2}\left[1-(x-1)^{2}\right]$ of the parabola. This example shows the link between elementary geometry (the definition of a parabola), simple analysis and algebra.

Example 3: Locus of the intersection point of the altitudes ( $S=h_{c} \cap h_{a}$ )
This is a simple standard example and it is possible that you can find it in a textbook. A calculation shows that $h_{c}: x=t, h_{a}: y=(1-t) x$. Thus by eliminating $t$ the equation $y=(1-x) x$ describes the movement of $S$ when point $C$ is moved with parameter $t$. It is easily possible to verify this result by DGS as well as by CAS.

## Example 4: Locus of the angle bisectors of a triangle ( $S=\omega_{\alpha} \cap \omega_{\beta}$ )

When one plots the curve of $S$ (again, $C$ is moved by changing parameter $t$ ) one could get the impression that it moves on an ellipse (Fig. 2). But simple numerical calculations of the halfaxis show that this impression is not true. In the following we solve this problem with a CAS. The usual approach is first to define the position vectors from A, B, C (Fig. 3). Then the unit vectors $\overrightarrow{\mathrm{AB}}, \overrightarrow{\mathrm{AC}}$ and $\overrightarrow{\mathrm{BC}}$ are constructed (Fig. 4, 5) and with their help $\mathrm{g}(\mathrm{r})$ and $\mathrm{h}(\mathrm{s})$ can be defined (Fig. 6) for the angle bisectors $\omega_{\alpha}$ and $\omega_{\beta}$. Subsequently we solve the equation $g(r)$ $=\mathrm{h}(\mathrm{s})$ with respect to the parameters r and $\mathrm{s}($ Fig. 7,8$)$ which leads to the parametric solution $\mathrm{x}(\mathrm{t})=\frac{\sqrt{\mathrm{t}^{2}+1}+\mathrm{t}}{\sqrt{(\mathrm{t}-1)^{2}+1}+\sqrt{\mathrm{t}^{2}+1}+1}$ and $\mathrm{y}(\mathrm{t})=\frac{1}{\sqrt{(\mathrm{t}-1)^{2}+1}+\sqrt{\mathrm{t}^{2}+1}+1}$ (Fig. 9)
It was a surprise that a CAS (TI-Nspire, Maple, Derive 5) does not eliminate $t$ from this parametric solution. Therefore we need to employ some brain cells in order to derive an explicit representation, e.g. something like $\mathrm{y}=\mathrm{f}(\mathrm{x})$.
We see that $\frac{x}{y}=\sqrt{t^{2}+1}+t$ and thus $\frac{y}{x}=\sqrt{t^{2}+1}-t$. From this we get $t=\frac{1}{2}\left(\frac{x}{y}-\frac{y}{x}\right)$
The insertion of this result into the parameterization along with some simplifications yields to $x^{2}(2 y-1)-x(2 y-1)-y^{2}=0$. Of course it is possible to check the result by plotting the curve and comparing it with the initial one.
This example shows that the use of DGS and CAS in some cases is not successful. But it shows that different representations and the switch between different approaches such as, plotting, calculating, using CAS, and ultimately using some "manual" math skills can do the trick.

## References:

[1] Moldenhauer, W. \& Zappe, W. (2008) Walking-tours. TI-Communications, 2, 27-31
[2] Zappe, W. \& Moldenhauer, W. \& Graubner, S. (2007) A locus of the intersection point of the angle bisectors of a triangle. TI-Communications, 1, 26-28
[3] Moldenhauer, W. \& Zappe, W. (2010) Loci. TI-Communications, 1, 27-31


Fig. 1 - The heart curve. Please note, the hearts have been colored in green because the author's state Thuringia is colloquially called "The green heart of Germany".


Fig. 2 - When plotted, S seems to move on an ellipse.

$$
\frac{\square\left[\begin{array}{l}
0 \\
0
\end{array}\right] \rightarrow a:\left[\begin{array}{l}
1 \\
0
\end{array}\right] \rightarrow b:\left[\begin{array}{l}
t \\
1
\end{array}\right] \rightarrow c}{\text { WINKHÁ1 }}
$$

Fig. 3 - Definition of position vectors


Fig. 4 - Definition of unit vectors $A B$ and $A C$


Fig. 5 - Definition of unit vector BC


Fig. 6 - Definition of the angle bisectors


IINW: Eereichserseb kann Sroker sein

Fig. 7 - Solution of $g(r)=h(s)$


Fig. 8 - Calculation of S


Fig. 9 - Obtained parametric solution for S

