TIME 2010, Malaga

Using Rational Arithmetic to Develop a Proof
"What Josef and Carl Saw"

Josef Böhm, Würmla, Austria and Carl Leinbach, Gardener, USA

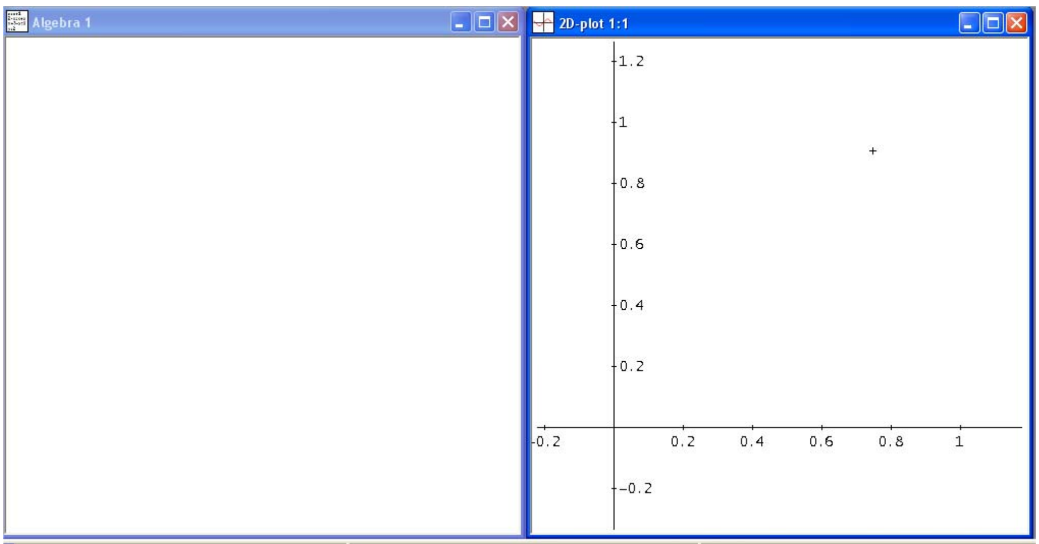
Josef, the publisher of the DERIVE & TI-CAS Newsletters revised the DNL#22 (originally dated from 1996) with one of Carl's and Marvin's Laboratory papers:

D-N-L#22	Carl's and Marvin's Laboratory 2	p29

Finding a Limit via Geometric Reasoning

Carl Leinbach and Marvin Brubaker, USA

Before we begin this investigation, adjust the graphics window to our needs using the Set > Aspect Ratio > 1:1 option. The screen should look similar to the figure below.



Consider the following sequence of points: $P_n = \begin{cases} [0,0] & n=0 \\ [0,1] & n=1 \\ [1,0] & n=2 \\ \frac{1}{2}P_{n-3} + \frac{1}{2}P_{n-2} & \text{otherwise} \end{cases}$

Notice that this sequence is defined recursively. *DERIVE* allows us to make recursive definitions. We use the IF statement.

```
P(n):=IF(n=0,[0, 0],IF(n=1,[0,1],IF(n=2,[1,0],1/2P(n-3)+1/2P(n-2))))
```

In this case we had to nest the IF statements three deep. That is because we had three special cases. This function, because of its recursive nature, is slow to evaluate for an n of any size, whatsoever. Nonetheless, author

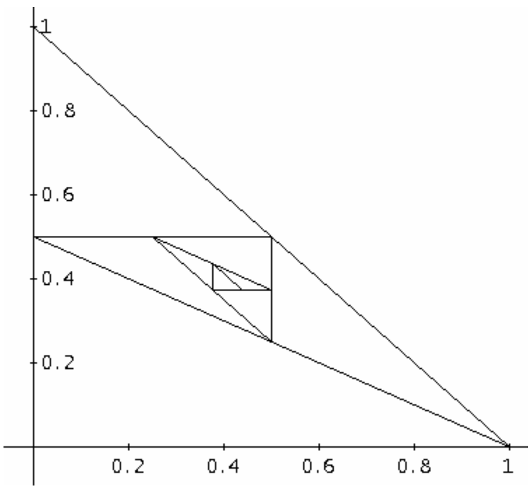
```
VECTOR(P(n),n,0,10)
```

and plot the sequence.

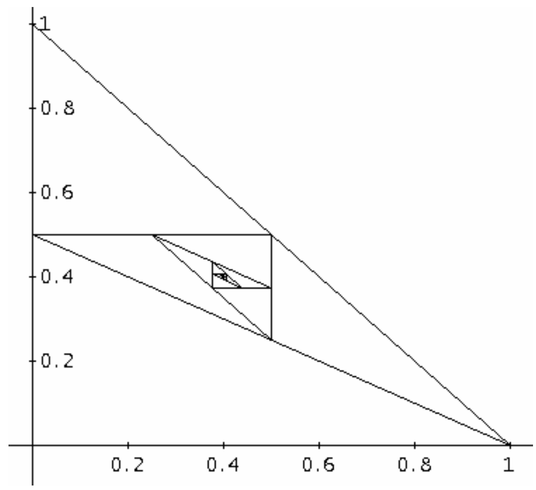
(It is not necessary to simplify the expression – giving a matrix of points. But take care that you have activated the Option > *Simplify before plotting* or *Approximate before plotting* in the plot window.

Set the Points in the Display Options *Connected* and *Size Small*.

The next figures show the evaluation of the first 10 terms of the sequence and also the first 20 terms. If we move the crosshair on the graph where the plot is dense, i.e., the point of apparent convergence we get a reading of approximately [0.4, 0.4].



`VECTOR(P(n),n,0,10)`

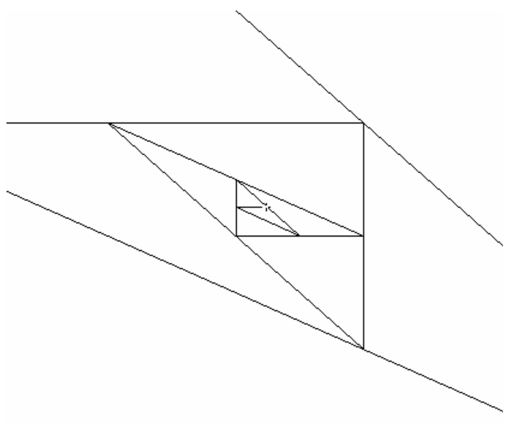


`VECTOR(P(n),n,0,20)`

We can zoom in and then we read off the coordinates of the crosshair [0.40029, 0.40042].

We can show the last term of the sequence given right above and we get a similar result: [0.40039…, 0.40039 …].

Of course, we had not proved any result. However, the visual evidence is convincing that a limit does exist ([0.4, 0.4]?) and we have a visual illustration of the process of convergence.



As Carl wrote, the recursive function is slow – try for n = 50! With *DERIVE* 5 and higher we can write a small program – without applying the interesting recursive function from above – which allows to calculate much more elements of this sequence.

Josef's comment was:

The challenge is still there: Proof that the limit is [0.4, 0.4]!

Josef wrote to Carl which was the beginning of an exchange of emails.

8 January 2010

Dear Carl,

I am now revising DNL#22 which contains Carl's and Marvins's Lab #2, "Finding a Limit via Geometric Reasoning".
I had to change some things due to the fact that DERIVE has changed a lot since 1996. I attach the revised contribution. Hope that you are satisfied with the new form (including a small program).

My question is: do you have a proof for the limit [2/5, 2/5]?

Best regards
Josef

11 January 2010

Josef -
I have not started on Lab 2, but hope to get to it before we leave on Wednesday morning. I have been working on meeting the (now revised) deadline before our Costa Rica trip. I enclosed the vastly revised paper in the hopes that you may find the example that I did on "Time Since Death" useful for your upcoming workshop. The referees wanted me to make my examples more "beefy", i.e. do some more substantial mathematics and involve the CAS more than I did in the original paper we submitted.

Dear Pat and Carl,
please don´t hurry - the proof is not so important. Enjoy your holidays.

12 January 2010

Josef -
While I was in the doctor's examination room waiting for the doctor to arrive, I tore off a piece of the paper covering the examination table and started to write out terms of the sequence. I got up to 16 terms.

5. Then prove that the lim(P(4*i)) = 2/5.

At the moment everything is based on my suppositions, not proven fact. I will keep working. Just wanted to keep you up to date.

some days later

Dear Carl,
Thanks for your efforts.
I am on a very similar way - to investigate the pattern of the numerators.

Hi Carl,
I attach my ideas for proving the limit.

27 January 2010

Josef -

I have attached the proof of the limit. I worked on it mainly on the plane ride to Costa Rica and a little bit during our visit to Costa Rica. It took a little more than I expected and as I note there is still one part that I want to clean up. I gave you an outline of that part. It is essential to the argument and I don't like the fact that it gets rather messy with the arithmetic.

Josef - 3 February 2010

I sent you this about a week ago and hadn't heard back. I was wondering what you thought. I think that it could make a good talk on combining the use of the rational arithmetic display of DERIVE to stimulate conjectures for solid mathematical analysis and then developing a proof. This is what we have been talking about for years. What do you think? BTW, I see that your though path and mine crossed at few crucial points. I was thinking that maybe we could develop a joint DNL article or a TIME talk on this type of use of DERIVE. Once again, what do you think?
-Carl

This was the first time when TIME 2010 and a possible common talk were mentioned! And Carl did not give up his idea:

8 February 2010

Josef I have mentioned a joint presentation at Malaga or a DNL article (your choice). Here is how I thought it could go:

History: The DNL #22 article attributed to Marvin and Carl; a request from Josef for an analytic proof of the limit

Observation Phase: Writing a brief program to examine terms of the sequence; the advantage of the rational arithmetic calculations and print out of DERIVE (and other CAS's)

Conjectures: What Josef saw (even though we worked independently, you were first); what Carl saw; putting conjectures to the test: Using mathematical induction to construct a proof

What do you think? I like the idea, because it uses a skill that we hope to develop amongst our students and uses CAS in much more than a "button pushing mode", which is what some of our antagonists accuse proponents of using CAS in teaching say we are professing.

16/18/20 February 2010

Josef -

Here is the promised draft of the Malaga presentation. Let me know what you think? Once we have the final form for the abstract, I will submit it.
-Carl

Dear Carl,

It looks good,; I am busy filling the gap(s) in my PROOF. Maybe that we could add one sentence about possible generalizations (changing the initial values, ...).

I attach a DERIVE file containing a general form for creating our sequence of points together with a nonrecursive way to create the sequence with the requested lim. Josef

8 March 2010

To time2010@ctima.uma.es

Please, find attached in this mail a (lecture or workshop) proposal for the (ACDCA strand) (TI-Nspire and Derive strand) (Please, indicate the appropriate format and strand).

This is a Lecture Proposal for the TI-Nspire & Derive Strand

Thank you,

Carl Leinbach

How Carl Attacked The Challenge

Let's suppose that a student had seen the Fibonacci sequence and the proof that the limit of the ratio of successive terms of that sequence converges to the "Golden Mean."

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$$

This approach simply can not be mimicked. It leads nowhere. **WHY?**

A next approach might be to try to visualize the terms of the sequence and look for some patterns. Suppose we try to familiarize ourselves with the nature of the sequence without using the features of a CAS, i.e. print out the decimal approximations to the sequence:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0.5 & 0.5 & 0.25 & 0.5 & 0.375 & 0.375 & 0.4375 & 0.375 & 0.40625 & 0.40625 & 0.390625 & 0.40625 & 0.3984375 & 0.3984375 & 0.40234375 & 0.3984375 & 0.400390625 & 0.400390625 & 0.3994140625 & 0.400390625 \\ 0 & 1 & 0 & 0.5 & 0.5 & 0.25 & 0.5 & 0.375 & 0.375 & 0.4375 & 0.375 & 0.40625 & 0.40625 & 0.390625 & 0.40625 & 0.3984375 & 0.3984375 & 0.40234375 & 0.3984375 & 0.400390625 & 0.400390625 & 0.3994140625 & 0.400390625 & 0.3999023437 \end{bmatrix}$$

What patterns do you see?

Here's What He Saw

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0.5 & 0.5 & 0.25 & 0.5 & 0.375 & 0.375 & 0.4375 & 0.375 & 0.40625 & 0.40625 & 0.390625 & 0.40625 & 0.3984375 & 0.3984375 & 0.40234375 & 0.3984375 & 0.400390625 & 0.400390625 & 0.3994140625 & 0.400390625 \\ 0 & 1 & 0 & 0.5 & 0.5 & 0.25 & 0.5 & 0.375 & 0.375 & 0.4375 & 0.375 & 0.40625 & 0.40625 & 0.390625 & 0.40625 & 0.3984375 & 0.3984375 & 0.40234375 & 0.3984375 & 0.400390625 & 0.400390625 & 0.3994140625 & 0.400390625 & 0.3999023437 \end{bmatrix}$$

Observation 1: Every term of the first sequence lags one term behind the second sequence. Thus, we really only need to deal with one sequence.

Proof: (Using the Principle of Mathematical induction)

Base Case: Look at the terms of the sequence printed out above

General Case: Assume the result holds for all $k < n$. Then

$$P_{n,1} = \frac{1}{2}\left(P_{n-3,1} + P_{n-2,1}\right) = \frac{1}{2}\left(P_{n-4,2} + P_{n-2,1}\right) \tag{1}$$

$$P_{n-1,2} = \frac{1}{2}\left(P_{n-4,2} + P_{n-3,2}\right) = \frac{1}{2}\left(P_{n-4,2} + P_{n-2,1}\right) \tag{2}$$

Where $P_{n,1}$ designates the n-th term in the first sequence and $P_{n,2}$ the same term in the second sequence. The second equality in both (1) and (2) are a result of the induction hypothesis.

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0.5 & 0.5 & 0.25 & 0.5 & 0.375 & 0.375 & 0.4375 & 0.375 & 0.40625 & 0.40625 & 0.390625 & 0.40625 & 0.3984375 & 0.3984375 & 0.40234375 & 0.3984375 & 0.400390625 & 0.400390625 & 0.3994140625 & 0.400390625 \\ 0 & 1 & 0 & 0.5 & 0.5 & 0.25 & 0.5 & 0.375 & 0.375 & 0.4375 & 0.375 & 0.40625 & 0.40625 & 0.390625 & 0.40625 & 0.3984375 & 0.3984375 & 0.40234375 & 0.3984375 & 0.400390625 & 0.400390625 & 0.3994140625 & 0.400390625 & 0.3999023437 \end{bmatrix}$$

Observation 2: $P_{4n,1} = P_{4n,2}$ for all $n = 0, 1, 2, 3, \ldots$

Proof: At the moment, it seems like the definition of the sequence is not going to get us to an obvious proof of this conjecture.

Let's see if something pops out by looking at the sequence in its rational number presentation. So let's turn to DERIVE:

```
     pts(n, pt) :=
       Prog
         pt := [0, 0; 0, 1; 1, 0]
         k := 4
#1:      Loop
           If k > n
              RETURN pt
           pt := APPEND(pt, [1/2·(pt↓(k - 3) + pt↓(k - 2))])
           k := k + 1

#2:  pts(24)'
```

#3:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} & \frac{7}{16} & \frac{3}{8} & \frac{13}{32} & \frac{13}{32} & \frac{25}{64} & \frac{13}{32} & \frac{51}{128} & \frac{51}{128} & \frac{103}{256} & \frac{51}{128} & \frac{205}{512} & \frac{205}{512} & \frac{409}{1024} & \frac{205}{512} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} & \frac{7}{16} & \frac{3}{8} & \frac{13}{32} & \frac{13}{32} & \frac{25}{64} & \frac{13}{32} & \frac{51}{128} & \frac{51}{128} & \frac{103}{256} & \frac{51}{128} & \frac{205}{512} & \frac{205}{512} & \frac{409}{1024} & \frac{205}{512} & \frac{819}{2048} \end{bmatrix}$$

Observation 3: $P_{4i-1,2} = P_{4i,2} = P_{4i+2,2}$ for all $i = 1,2,3,\ldots$

Proof: Assume that the result holds for all $k < i$

$$P_{4i-1,2} = \frac{1}{2}\left(P_{4i-4,2} + P_{4i-3,2}\right) = \frac{1}{2}\left(P_{4(i-1),2} + P_{4i-3,2}\right) = \frac{1}{2}\left(P_{4(i-1)+2,2} + P_{4i-3,2}\right) =$$
$$= \frac{1}{2}\left(P_{4i-2,2} + P_{4i-3,2}\right) = P_{4i,2}$$

by definition of the recursive sequence.

The next to last equality was a result of the induction hypothesis.

Finally, $P_{4i+2,2} = \frac{1}{2}\left(P_{4i-2,2} + P_{4i,2}\right) = P_{4i,2}$ by the sequence definition and the first part of this proof.

If we combine Observation 1 and Observation 3 we have the proof for Observation 2. Thus, the part of the "Geometric Reasoning" that states that the limit of the sequence of points lies on the line $y = x$ is indeed correct.

But:

What is the value of the limit?

Finding the Limit of $\{P_{4i,1}\}$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} & \frac{7}{16} & \frac{3}{8} & \frac{13}{32} & \frac{13}{32} & \frac{25}{64} & \frac{13}{32} & \frac{51}{128} & \frac{51}{128} & \frac{103}{256} & \frac{51}{128} & \frac{205}{512} & \frac{205}{512} & \frac{409}{1024} & \frac{205}{512} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} & \frac{7}{16} & \frac{3}{8} & \frac{13}{32} & \frac{13}{32} & \frac{25}{64} & \frac{13}{32} & \frac{51}{128} & \frac{51}{128} & \frac{103}{256} & \frac{51}{128} & \frac{205}{512} & \frac{205}{512} & \frac{409}{1024} & \frac{205}{512} & \frac{819}{2048} \end{bmatrix}$$

Finding this limit and then invoking observation 3 and one more observation, we can easily use a classic ε, δ proof to show that the limit of this subsequence is the limit of the entire sequence.

Looking at the sequence of first coordinates, we see that the even terms for $i \geq 2$ (remember I call the first term P_0) have successive powers of two in the denominator . Here is a DERIVE program and its result to look at this sequence:

```
Even_Terms(n, ev) :=
  Prog
    pt := [0, 0; 0, 1; 1, 0]
    ev := [0, 1]
    k := 4
    Loop
      If k > n
         RETURN ev
      pt := APPEND(pt, [1/2·(pt↓(k - 3) + pt↓(k - 2))])
      If MOD(k, 2) = 1
         ev := APPEND(ev, [pt↓k↓1])
      k := k + 1

Even_Terms(35)
```

$$\left[0,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{3}{8},\ \frac{7}{16},\ \frac{13}{32},\ \frac{25}{64},\ \frac{51}{128},\ \frac{103}{256},\ \frac{205}{512},\ \frac{409}{1024},\ \frac{819}{2048},\ \frac{1639}{4096},\ \frac{3277}{8192},\ \frac{6553}{16384},\ \frac{13107}{32768},\ \frac{26215}{65536}\right]$$

Observation 4: $P_{2i,1} = P_{2(i-1),1} + \frac{(-1)^{\left\lfloor \frac{i}{2} \right\rfloor}}{2^{i-1}}$ for all $i = 1, 2, 3, \ldots$ and $\left\lfloor \frac{i}{2} \right\rfloor$ denotes the floor function.

Proof: Once again, we will assume that the result holds for all $k < i$.

We go back to the basic definition for the basic definition sequence, P_n, and work from there.

$$P_{2i,1} = \frac{1}{2}\left(P_{2i-3,1} + P_{2i-1,1}\right) = \frac{1}{2}\left(P_{2i-3,1} + P_{2(i-1),1}\right)$$

This argument is laden with notation and not terribly instructive, so let's give only an overview of how it goes:

Break the attack into two cases: i even and i odd i.e. $2i$ a multiple of 4 and not a multiple of 4. It is really the first case that we want, but need to prove it for all even terms. Basically, Observations 1 and 3 get the $P_{2i-3,1}$ term above to a previous multiple of 4 and then we work back up. The arithmetic gets messy and the exponents are a little hard to handle, but it eventually all works out. Note that the sign change always takes place at the multiples of 4. As was mentioned: Observations 1 & 3 are the keys.

Observation 5: $P_{4i,1} = P_{4(i-1),1} + \frac{(-1)^{i-1}}{2 \cdot 4^i}$ for $i = 1, 2, 3, \ldots$,and thus,

$$P_{4i,1} = \sum_{k=1}^{i} \frac{(-1)^{k-1}}{2 \cdot 4^k} = \frac{1}{2} \sum_{k=1}^{i} \frac{(-1)^{k-1}}{4^k}.$$

Proof: This is just a matter of extracting the terms from Observation 4.

Observation 6: $\lim_{n\to\infty} P_{4n,1} = \frac{2}{5}$ and, thus, $\lim_{n\to\infty} P_{4n,2} = \frac{2}{5}$.

Proof: We turn this one over to DERIVE: $\frac{1}{2} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} = \frac{2}{5}$

Finally, we need only show that the sequences of first and second coordinates converge. We show that they are Cauchy Sequences of Real Numbers and use the fact that the Real Numbers are a complete metric space, i.e. all Cauchy Sequences converge.

Observation 7: The sequences $\{P_{n,1}\}$ and $\{P_{n,2}\}$ are Cauchy Sequences.

Proof: Let $\varepsilon > 0$, Observations1, 3, and 4 have shown that for any two adjacent terms in the interval from $4i$ to $4(i+1)$ the absolute value of the differences are: $0, \frac{1}{2^i}, \frac{1}{2^i}, \frac{1}{2^{i+1}}$, respectively. Take the largest of these differences, $\frac{1}{2^i}$, and say that $\left|P_{4\left[\frac{n}{4}\right],1} - \frac{2}{5}\right| < \frac{\varepsilon}{4}$.

Now, choose N such that for $n > N$, $\frac{1}{2^{\left[\frac{n}{4}\right]}} < \frac{\varepsilon}{4}$ and $\left|P_{4\left[\frac{n}{4}\right],1} - \frac{2}{5}\right| < \frac{\varepsilon}{4}$ then if $m, n > N$ we have

$$\left|P_{n,1} - P_{m,1}\right| = \left|\left(P_{n,1} - P_{4\left[\frac{n}{4}\right],1}\right) - \left(P_{m,1} - P_{4\left[\frac{m}{4}\right],1}\right) + \left(P_{4\left[\frac{n}{4}\right],1} - \frac{2}{5}\right) - \left(P_{4\left[\frac{m}{4}\right],1} - \frac{2}{5}\right)\right| \le \varepsilon$$

Thus, $P_{n,1}$ is a Cauchy Sequence and hence converges to the same limit as $P_{4i,1}$.

The sequence $P_{4i,2}$ is just one term ahead of $P_{4i,1}$ and, thus, also converges to $\frac{2}{5}$.

Carl Is Finally Finished!

How Josef Attacked The Challenge

<u>My first approach:</u>

This was the function for creating the visualisation, giving a sequence of points:

```
P(n) :=
  If n = 0
     [0, 0]
     If n = 1
        [0, 1]
        If n = 2
           [1, 0]
           1/2·P(n - 3) + 1/2·P(n - 2)

VECTOR(P(n), n, 0, 20)'
```

$$\begin{bmatrix} 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} & \frac{7}{16} & \frac{3}{8} & \frac{13}{32} & \frac{13}{32} & \frac{25}{64} & \frac{13}{32} & \frac{51}{128} & \frac{51}{128} & \frac{103}{256} & \frac{51}{128} & \frac{205}{512} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} & \frac{7}{16} & \frac{3}{8} & \frac{13}{32} & \frac{13}{32} & \frac{25}{64} & \frac{13}{32} & \frac{51}{128} & \frac{51}{128} & \frac{103}{256} & \frac{51}{128} & \frac{205}{512} & \frac{205}{512} \end{bmatrix}$$

I inspected the (sorted) numerators of the 1st components (the x-values) in order to find the pattern:

Because of the recursive nature of the definition it needs a long calculation time to find the list of the first 100 numerators!! The next function works iterative and is much much faster:

```
pts(n, m := 1/2, pt) :=
  Prog
    pt := [0, 0; 0, 1; 1, 0]
    k := 4
    Loop
      If k > n
         RETURN pt
      pt := APPEND(pt, [m·(pt↓(k - 3) + pt↓(k - 2))])
      k :+ 1
```

$$\text{pts(10)}' = \begin{bmatrix} 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} & \frac{7}{16} \end{bmatrix}$$

```
SORT(VECTOR(NUMERATOR(k), k, (pts(60))↓↓1))

[0, 0, 0, 1, 1, 1, 1, 1, 3, 3, 3, 7, 13, 13, 13, 25, 51, 51, 51, 103, 205, 205, 205, 409, 819, 819,
  819, 1639, 3277, 3277, 3277, 6553, 13107, 13107, 13107, 26215, 52429, 52429, 52429, 104857,
  209715, 209715, 209715, 419431, 838861, 838861, 838861, 1677721, 3355443, 3355443, 3355443,
  6710887, 13421773, 13421773, 13421773, 26843545, 53687091, 53687091, 53687091, 107374183]
```

which is without counting repeated appearances:

[0, 1, 3, 7, 13, 25, 51, 103, 205, 409, 819, 1639, 3277, 6553, 13107, 26215, 52429, 104857, 209715, 419431, 838861, 1677721, ….]

Starting with 7 we have always a package of 4 values containing the first and then three times the next value. I investigated the sequence of values from above starting with $n = 4$ which gives element 7:

n=4	7 =	$2^2 + 3$	
n=5	13 =	$2\cdot7-1$ =	$2^3+2\cdot3-1$
n=6	25 =	$2\cdot13-1$ =	$2^4+2^2\cdot3-3$
n=7	51 =	$2\cdot25+1$ =	$2^5+2^3\cdot3-2\cdot3+1$
n=8	103 =	$2\cdot51+1$ =	$2^6+2^4\cdot3-2^2\cdot3+3$
n=9	205 =	$2\cdot103-1$ =	$2^7+2^5\cdot3-2^3\cdot3+2\cdot3-1$
n=10	409 =	$2\cdot205-1$ =	$2^8+2^6\cdot3-2^4\cdot3+2^2\cdot3-3$
n=11	819 =	$2\cdot409+1$ =	$2^9+2^7\cdot3-2^5\cdot3+2^3\cdot3-2\cdot3+1$
n=12	1639 =	$2\cdot819+1$ =	$2^{10}+2^8\cdot3-2^6\cdot3+2^4\cdot3-2^2\cdot3+3$

…
…

The elements of the sequence formed by the first row of $P(n)$ from above are the numerators divided by 2^n.

I start with the elements with n = 4, 8, 12, … and try finding a general formula for the numerators

This is the "funny part" of the problem!!

$$\#1: \quad 2^{4\cdot i - 2} + 3\cdot \sum_{k=0}^{i-1} 2^{4\cdot k} - 3\cdot 2^{2} \cdot \sum_{k=0}^{i-2} 2^{4\cdot k}$$

$$\#2: \quad \frac{2^{4\cdot i + 1}}{5} + \frac{3}{5}$$

$$\#3: \quad \text{VECTOR}\left(\frac{\frac{2^{4\cdot i + 1}}{5} + \frac{3}{5}}{2^{4\cdot i}}, i, 0, 10\right)$$

$$\#4: \quad \left[1, \frac{7}{16}, \frac{103}{256}, \frac{1639}{4096}, \frac{26215}{65536}, \frac{419431}{1048576}, \frac{6710887}{16777216}, \frac{107374183}{268435456}, \frac{1717986919}{4294967296}, \frac{27487790695}{68719476736}, \frac{439804651111}{1099511627776}\right]$$

$$\#5: \quad \frac{\frac{2^{4\cdot i + 1}}{5} + \frac{3}{5}}{2^{4\cdot i}} = \frac{3\cdot 2^{-4\cdot i}}{5} + \frac{2}{5}$$

$$\#6: \quad \lim_{i\to\infty}\left(\frac{3\cdot 2^{-4\cdot i}}{5} + \frac{2}{5}\right) = \frac{2}{5}$$

Derive simplifies expression #1 to a nice formula. Applying VECTOR I can check the correctness of expression #5 and in the last step the limit of the partial sequence is given (even without a CAS).

I repeat the procedure for elements with $n = 5, 9, 13, \ldots$ and end again with the limit $\frac{2}{5}$.

$$\#7: \quad 2^{4\cdot i - 1} + 3\cdot 2\cdot \sum_{k=0}^{i-1} 2^{4\cdot k} - 3\cdot 2^{3} \cdot \sum_{k=0}^{i-2} 2^{4\cdot k} - 1$$

I can proceed in a similar way for the remaining elements of the sequence.

For $n = 6, 10, 14, \ldots$

$$2^{4\cdot i} + 3\cdot 2^{2} \cdot \sum_{k=0}^{i-1} 2^{4\cdot k} - 3\cdot \sum_{k=0}^{i-1} 2^{4\cdot k}$$

And finally for $n = 3, 7, 11, 15, \ldots$

$$\text{VECTOR}\left(\frac{2}{5}-\frac{2^{-4\cdot i-3}}{5},\ i,\ 0,\ 10\right)=\left[\frac{3}{8},\ \frac{51}{128},\ \frac{819}{2048},\ \frac{13107}{32768},\ \frac{209715}{524288},\ \frac{3355443}{8388608},\ \frac{53687091}{134217728},\ \frac{858993459}{2147483648},\ \frac{13743895347}{34359738368},\ \frac{219902325555}{549755813888},\ \frac{3518437208883}{8796093022208}\right]$$

$$\lim_{i\to\infty}\left(\frac{2}{5}-\frac{2^{-4\cdot i-3}}{5}\right)=\frac{2}{5}$$

All partial sequences tend to the same limit, so the limit is $\frac{2}{5}$.

I must admit that I was not really satisfied with my "PROOF", be cause I could not show that the pattern of the numerators and of the fractions will remain the same until infinity.

Inspired by Carl´s PROOF and by the fact that only natural numbers are involved I was quite sure that Proof by Induction must be the right "recipe"!

My next approach:

I used my formulae from above for generating a list of all fractions appearing in the sequence:

$$\#4:\quad p(k):=\left[\frac{2^{-k}}{5}+\frac{2}{5},\ \frac{2}{5}-\frac{3\cdot 2^{-k-1}}{5},\ \frac{2}{5}-\frac{2^{-k-2}}{5},\ \frac{3\cdot 2^{-k-3}}{5}+\frac{2}{5}\right]$$

$$\#5:\quad \text{VECTOR}(p(k),\ k,\ 1,\ 21,\ 4)=\begin{bmatrix}\frac{1}{2} & \frac{1}{4} & \frac{3}{8} & \frac{7}{16}\\ \frac{13}{32} & \frac{25}{64} & \frac{51}{128} & \frac{103}{256}\\ \frac{205}{512} & \frac{409}{1024} & \frac{819}{2048} & \frac{1639}{4096}\\ \frac{3277}{8192} & \frac{6553}{16384} & \frac{13107}{32768} & \frac{26215}{65536}\\ \frac{52429}{131072} & \frac{104857}{262144} & \frac{209715}{524288} & \frac{419431}{1048576}\\ \frac{838861}{2097152} & \frac{1677721}{4194304} & \frac{3355443}{8388608} & \frac{6710887}{16777216}\end{bmatrix}$$

We can see the table of the first 24 different fractions appearing in the sequence:

I started from the very beginning:

Then I came back to the original sequences of the 1st and 2nd components. My consideration was that both components are created in the same way, then I could stick to only one of them and I chose the x-coordinate. Function `pts(n)` returns the first n first coordinates of the sequence of points.

```
#4:   pts(40)'
```

#5:

$$\left[\begin{array}{cccccccccccccccccccc} 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} & \frac{7}{16} & \frac{3}{8} & \frac{13}{32} & \frac{13}{32} & \frac{25}{64} & \frac{13}{32} & \frac{51}{128} & \frac{51}{128} & \frac{103}{256} & \frac{51}{128} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} & \frac{7}{16} & \frac{3}{8} & \frac{13}{32} & \frac{13}{32} & \frac{25}{64} & \frac{13}{32} & \frac{51}{128} & \frac{51}{128} & \frac{103}{256} & \frac{51}{128} & \frac{205}{512} \end{array}\right.$$

$$\begin{array}{ccccccccccccc} \frac{205}{512} & \frac{205}{512} & \frac{409}{1024} & \frac{205}{512} & \frac{819}{2048} & \frac{819}{2048} & \frac{1639}{4096} & \frac{819}{2048} & \frac{3277}{8192} & \frac{3277}{8192} & \frac{6553}{16384} & \frac{3277}{8192} & \frac{13107}{32768} \\ \frac{205}{512} & \frac{409}{1024} & \frac{205}{512} & \frac{819}{2048} & \frac{819}{2048} & \frac{1639}{4096} & \frac{819}{2048} & \frac{3277}{8192} & \frac{3277}{8192} & \frac{6553}{16384} & \frac{3277}{8192} & \frac{13107}{32768} & \frac{13107}{32768} \end{array}$$

$$\left.\begin{array}{ccccccc} \frac{13107}{32768} & \frac{26215}{65536} & \frac{13107}{32768} & \frac{52429}{131072} & \frac{52429}{131072} & \frac{104857}{262144} & \frac{52429}{131072} \\ \frac{26215}{65536} & \frac{13107}{32768} & \frac{52429}{131072} & \frac{52429}{131072} & \frac{104857}{262144} & \frac{52429}{131072} & \frac{209715}{524288} \end{array}\right]$$

I prepared a tool:

I wanted to address each single element of the sequence, used the formulae p(k) from above and took in account the fact that it is better to consider packages of eight elements in a row instead of only four.

```
       el(n) :=
         If n ≤ 4
            [0, 0, 1, 0]↓n
            If MOD(n, 8) = 5 ∨ MOD(n, 8) = 6 ∨ MOD(n, 8) = 0
               (p(4·FLOOR((n + 3)/8) - 3))↓1
#6:            If MOD(n, 8) = 1 ∨ MOD(n, 8) = 2 ∨ MOD(n, 8) = 4
                  (p(4·FLOOR((n + 3)/8) - 3))↓3
                  If MOD(n, 8) = 7
                     (p(4·FLOOR((n + 3)/8) - 3))↓2
                     (p(4·FLOOR((n + 3)/8) - 3))↓4
```

```
#7:   VECTOR(el(k), k, 41)
```

#8:

$$\left[0,\ 0,\ 1,\ 0,\ \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, \frac{7}{16}, \frac{3}{8}, \frac{13}{32}, \frac{13}{32}, \frac{25}{64}, \frac{13}{32}, \frac{51}{128}, \frac{51}{128}, \frac{103}{256}, \right.$$

$$\frac{51}{128}, \frac{205}{512}, \frac{205}{512}, \frac{409}{1024}, \frac{205}{512}, \frac{819}{2048}, \frac{819}{2048}, \frac{1639}{4096}, \frac{819}{2048}, \frac{3277}{8192}, \frac{3277}{8192}, \frac{6553}{16384},$$

$$\left. \frac{3277}{8192}, \frac{13107}{32768}, \frac{13107}{32768}, \frac{26215}{65536}, \frac{13107}{32768}, \frac{52429}{131072}, \frac{52429}{131072}, \frac{104857}{262144}, \frac{52429}{131072}, \frac{209715}{524288}\right]$$

Compare with the first row of #5 from above.

Generalization of the problem

For keeping the procedure more general I introduce the matrix **ini** which is matrix defined by points #2 and #3; the first point is the origin by default.

```
      ptss(n, ini, pt, m := 1/2) :=
        Prog
          pt := APPEND([[0, 0]], ini)
          k := 4
#9:       Loop
            If k > n
               RETURN pt
            pt := APPEND(pt, [m·(pt↓(k - 3) + pt↓(k - 2))])
            k :+ 1
```

As the first and second coordinates are following the same rule, it is sufficient to investigate only one of them. I am choosing the *x*-coordinates.

Let it keep as general as possible (m = ½ by default):

#10: $$\text{ptss}\left(41, \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\right)\downarrow_1$$

#11: $$\left[0, x_1, x_2, \frac{x_1}{2}, \frac{x_2 + x_1}{2}, \frac{2 \cdot x_2 + x_1}{4}, \frac{x_2 + 2 \cdot x_1}{4}, \frac{4 \cdot x_2 + 3 \cdot x_1}{8}, \frac{3 \cdot (x_2 + x_1)}{8}, \frac{6 \cdot x_2 + 7 \cdot x_1}{16},\right.$$
$$\frac{7 \cdot x_2 + 6 \cdot x_1}{16}, \frac{12 \cdot x_2 + 13 \cdot x_1}{32}, \frac{13 \cdot (x_2 + x_1)}{32}, \frac{26 \cdot x_2 + 25 \cdot x_1}{64}, \frac{25 \cdot x_2 + 26 \cdot x_1}{64}, \frac{52 \cdot x_2 + 51 \cdot x_1}{128},$$
$$\frac{51 \cdot (x_2 + x_1)}{128}, \frac{102 \cdot x_2 + 103 \cdot x_1}{256}, \frac{103 \cdot x_2 + 102 \cdot x_1}{256}, \frac{204 \cdot x_2 + 205 \cdot x_1}{512}, \frac{205 \cdot (x_2 + x_1)}{512}, \frac{410 \cdot x_2 + 409 \cdot x_1}{1024},$$
$$\frac{409 \cdot x_2 + 410 \cdot x_1}{1024}, \frac{820 \cdot x_2 + 819 \cdot x_1}{2048}, \frac{819 \cdot (x_2 + x_1)}{2048}, \frac{1638 \cdot x_2 + 1639 \cdot x_1}{4096}, \frac{1639 \cdot x_2 + 1638 \cdot x_1}{4096},$$
$$\frac{3276 \cdot x_2 + 3277 \cdot x_1}{8192}, \frac{3277 \cdot (x_2 + x_1)}{8192}, \frac{6554 \cdot x_2 + 6553 \cdot x_1}{16384}, \frac{6553 \cdot x_2 + 6554 \cdot x_1}{16384}, \frac{13108 \cdot x_2 + 13107 \cdot x_1}{32768},$$
$$\frac{13107 \cdot (x_2 + x_1)}{32768}, \frac{26214 \cdot x_2 + 26215 \cdot x_1}{65536}, \frac{26215 \cdot x_2 + 26214 \cdot x_1}{65536}, \frac{52428 \cdot x_2 + 52429 \cdot x_1}{131072}, \frac{52429 \cdot (x_2 + x_1)}{131072},$$

For me it is important to double check the single steps of the procedure:

Substituting [0,1] for x = [x1,x2] results in the coefficients of x2 which is the list of the 1st coordinates of the points:

$$\#12: \left[0,\ 0,\ 1,\ 0,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{8},\ \frac{3}{8},\ \frac{7}{16},\ \frac{3}{8},\ \frac{13}{32},\ \frac{13}{32},\ \frac{25}{64},\ \frac{13}{32},\ \frac{51}{128},\ \frac{51}{128},\ \frac{103}{256},\ \frac{51}{128},\ \frac{205}{512},\ \frac{205}{512},\ \frac{409}{1024},\ \frac{205}{512},\ \frac{819}{2048},\ \frac{819}{2048},\ \frac{1639}{4096},\ \frac{819}{2048},\ \frac{3277}{8192},\ \frac{3277}{8192},\ \frac{6553}{16384},\ \frac{3277}{8192},\ \frac{13107}{32768},\ \frac{13107}{32768},\ \frac{26215}{65536},\ \frac{13107}{32768},\ \frac{52429}{131072},\ \frac{52429}{131072},\ \frac{104857}{262144},\ \frac{52429}{131072},\ \frac{209715}{524288}\right]$$

I substitute [1,0] for x = [x1,x2] for obtaining the coefficients of x1 (= 2nd coordinates of the points):

$$\#13: \left[0,\ 1,\ 0,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{8},\ \frac{3}{8},\ \frac{7}{16},\ \frac{3}{8},\ \frac{13}{32},\ \frac{13}{32},\ \frac{25}{64},\ \frac{13}{32},\ \frac{51}{128},\ \frac{51}{128},\ \frac{103}{256},\ \frac{51}{128},\ \frac{205}{512},\ \frac{205}{512},\ \frac{409}{1024},\ \frac{205}{512},\ \frac{819}{2048},\ \frac{819}{2048},\ \frac{1639}{4096},\ \frac{819}{2048},\ \frac{3277}{8192},\ \frac{3277}{8192},\ \frac{6553}{16384},\ \frac{3277}{8192},\ \frac{13107}{32768},\ \frac{13107}{32768},\ \frac{26215}{65536},\ \frac{13107}{32768},\ \frac{52429}{131072},\ \frac{52429}{131072},\ \frac{104857}{262144},\ \frac{52429}{131072},\ \frac{209715}{524288},\ \frac{209715}{524288}\right]$$

These are the coefficients of x_1 and x_2 respectively:

```
#14:   x2_p(n) := el(n)

#15:   x1_p(n) := el(n + 1)
```

I split the fractions into their summands try to proof the pattern of the coefficients by induction.

Assume that the rule is valid until element x1_p(n) with mod(n,8) = 0; we would like to find element x1_p(n+1) by

1/2 *(x1_p(n+1-3)+x1_p(n+1-2)) = 1/2*(x1_p(n-2)+x1_p(n-1)).

Then x1_p(n-2) with mod(n-2,8) = 6 and x1_p(n-1) with mod(n-1,8) = 7 will - hopefully - give x1_p(n+1) with mod(n+1,8) = 1.

First of all I check for n = 40:

$$\#16: \quad \text{x1\_p}(40) = \frac{209715}{524288}$$

$$\#17: \quad \text{x1\_p}(41) = \frac{1}{2}\cdot(\text{x1\_p}(38) + \text{x1\_p}(39))$$

$$\#18: \quad \frac{209715}{524288} = \frac{209715}{524288}$$

I copied function el(n) because I am needing the subexpressions for the different cases of mod(n,8).

```
       el(n) :=
         If n ≤ 4
            [0, 0, 1, 0]↓n
            If MOD(n, 8) = 5 ∨ MOD(n, 8) = 6 ∨ MOD(n, 8) = 0
               (p(4·FLOOR((n + 3)/8) - 3))↓1
#19:           If MOD(n, 8) = 1 ∨ MOD(n, 8) = 2 ∨ MOD(n, 8) = 4
                  (p(4·FLOOR((n + 3)/8) - 3))↓3
                  If MOD(n, 8) = 7
                     (p(4·FLOOR((n + 3)/8) - 3))↓2
                     (p(4·FLOOR((n + 3)/8) - 3))↓4
```

Then x1_p(n-2): (mod(n,8) = 6)

#20: $\mathrm{SUBST}\left(\left(p\left(4\cdot \mathrm{FLOOR}\left(\frac{n+3}{8}\right)-3\right)\right)_1,\ n,\ n-2\right)$

#21: $\frac{2\cdot 2^{3-4\cdot \mathrm{FLOOR}(n/8+1/8)}}{5}+\frac{2}{5}$

#22: $\mathrm{VECTOR}\left(\frac{2\cdot 2^{3-4\cdot \mathrm{FLOOR}(n/8+1/8)}}{5}+\frac{2}{5},\ n,\ 8,\ 48,\ 8\right)=\left[\frac{1}{2},\ \frac{13}{32},\ \frac{205}{512},\ \frac{3277}{8192},\ \frac{52429}{131072},\ \frac{838861}{2097152}\right]$

Expression #22 are elements #6, 14, 22, 30, …

x1_p(n-1): (mod(n,8) = 7)

#23: $\mathrm{SUBST}\left(\left(p\left(4\cdot \mathrm{FLOOR}\left(\frac{n+3}{8}\right)-3\right)\right)_2,\ n,\ n-1\right)$

#24: $\frac{2}{5}-\frac{2^2\cdot 3\cdot 2^{2-4\cdot \mathrm{FLOOR}(n/8+1/4)}}{5}$

#25: $\mathrm{VECTOR}\left(\frac{2}{5}-\frac{2^2\cdot 3\cdot 2^{2-4\cdot \mathrm{FLOOR}(n/8+1/4)}}{5},\ n,\ 8,\ 48,\ 8\right)=\left[\frac{1}{4},\ \frac{25}{64},\ \frac{409}{1024},\ \frac{6553}{16384},\ \frac{104857}{262144},\ \frac{1677721}{4194304}\right]$

Expression #25 are elements #7, 15, 23, 31, …

This is - should be - the next element in the sequence x1_p(n+1): (mod(n,8) = 1)

#26: $\mathrm{SUBST}\left(\left(p\left(4\cdot \mathrm{FLOOR}\left(\frac{n+3}{8}\right)-3\right)\right)_3,\ n,\ n+1\right)$

#27: $\frac{2}{5}-\frac{2\cdot 2^{1-4\cdot \mathrm{FLOOR}(n/8+1/2)}}{5}$

#28: $\mathrm{VECTOR}\left(\frac{2}{5}-\frac{2\cdot 2^{1-4\cdot \mathrm{FLOOR}(n/8+1/2)}}{5},\ n,\ 8,\ 48,\ 8\right)=\left[\frac{3}{8},\ \frac{51}{128},\ \frac{819}{2048},\ \frac{13107}{32768},\ \frac{209715}{524288},\ \frac{3355443}{8388608}\right]$

The next check holds:

$$\#29:\quad \frac{1}{2}\cdot\left(\left[\frac{1}{2}, \frac{13}{32}, \frac{205}{512}, \frac{3277}{8192}, \frac{52429}{131072}, \frac{838861}{2097152}\right] + \left[\frac{1}{4}, \frac{25}{64}, \frac{409}{1024}, \frac{6553}{16384}, \frac{104857}{262144}, \frac{1677721}{4194304}\right]\right)$$

$$\#30:\quad \left[\frac{3}{8}, \frac{51}{128}, \frac{819}{2048}, \frac{13107}{32768}, \frac{209715}{524288}, \frac{3355443}{8388608}\right]$$

Now follows the interesting step: 1/2*(#21 + #24) = #27 ??

$$\#31:\quad \frac{1}{2}\cdot\left(\frac{2^3\cdot 2^{-4\cdot\mathrm{FLOOR}(n/8 + 1/8)}}{5} + \frac{2}{5} + \left(\frac{2}{5} - \frac{2^2\cdot 3\cdot 2^{-4\cdot\mathrm{FLOOR}(n/8 + 1/4)}}{5}\right)\right)$$

$$\#32:\quad \frac{2^2\cdot 2^{-4\cdot\mathrm{FLOOR}(n/8 + 1/8)}}{5} - \frac{2^1\cdot 3\cdot 2^{-4\cdot\mathrm{FLOOR}(n/8 + 1/4)}}{5} + \frac{2}{5}$$

DERIVE does not simplify further because it has no information about the nature of n.

But we have: n is divisible by 8 (mod(n,8) = 0).

We know that: for all n with mod(n,8) = 0: floor(n/8 + 1/8) = floor(n/8 + 1/4) = n/8.

$$\#33:\quad \frac{2^2\cdot 2^{-4\cdot(n/8)}}{5} - \frac{2^1\cdot 3\cdot 2^{-4\cdot(n/8)}}{5} + \frac{2}{5}$$

$$\#34:\quad \frac{2}{5} - \frac{2^{(2 - n)/2}}{5}$$

We can do that with the CAS, too:

$$\frac{1}{2}\cdot\left(\frac{2^3\cdot 2^{-4\cdot\mathrm{FLOOR}(n/8 + 1/8)}}{5} + \frac{2}{5} + \left(\frac{2}{5} - \frac{2^2\cdot 3\cdot 2^{-4\cdot\mathrm{FLOOR}(n/8 + 1/4)}}{5}\right)\right)$$

$$\frac{1}{2}\cdot\left(\frac{2^3\cdot 2^{-4\cdot\mathrm{FLOOR}(8\cdot n\_/8 + 1/8)}}{5} + \frac{2}{5} + \left(\frac{2}{5} - \frac{2^2\cdot 3\cdot 2^{-4\cdot\mathrm{FLOOR}(8\cdot n\_/8 + 1/4)}}{5}\right)\right)$$

$$\frac{2^2\cdot 2^{-4\cdot\mathrm{FLOOR}(n\_ + 1/8)}}{5} - \frac{2^1\cdot 3\cdot 2^{-4\cdot\mathrm{FLOOR}(n\_ + 1/4)}}{5} + \frac{2}{5}$$

$n\_ \in \mathrm{Integer}\ (0, \infty)$

$$\frac{2}{5} - \frac{2^{1 - 4\cdot n\_}}{5}$$

$$\frac{2}{5} - \frac{2^{1 - 4\cdot(n/8)}}{5}$$

$$\frac{2}{5} - \frac{2^{(2-n)/2}}{5}$$

We “simplify” expression #27 in the same way:

$$\#27: \quad \frac{2}{5} - \frac{2^{1}\cdot 2^{-4\cdot \mathrm{FLOOR}(n/8 + 1/2)}}{5}$$

The last steps are easy work:

$$\#35: \quad \frac{2}{5} - \frac{2^{1}\cdot 2^{-4\cdot \mathrm{FLOOR}(n/8 + 1/2)}}{5}$$

$$\#36: \quad \frac{2}{5} - \frac{2^{1}\cdot 2^{-4\cdot (n/8)}}{5}$$

$$\#37: \quad \frac{2}{5} - \frac{2^{(2-n)/2}}{5}$$

#34 = #37, which was to be proofed.

We can repeat the procedure for all cases and proof show the identities of x1_p(n+1) = 1/2·(x1_p(n-2)+x1_p(n-1)) for all positions of n within a package of 8 in a row.

It is obvious that for the second part of the x-value = x2_p(n) the proof will also hold.

What we also can see is the fact that the full x-value will be 2/5·x1 + 2/5·x2 + f1(n)·x1 + f2(n)·x2 where f1 and f2 are functions with 2^n in the denominator. The same is happening with the y-values.

Calculating the limits, the functions are tending to 0 and the limit of the sequence of points with [x0,y0] = [0,0] will end in [2/5·(x1+x2), 2/5·(y1+y2)].

See an example: Initial points are [0,0], [5,-4] and [11,9].

If my idea will hold then the sequence should end in [2/5·(5+11), 2/5·(-4+9)] = [6.4, 2].

$$\#47: \quad \left(\mathrm{ptss}\left(100, \begin{bmatrix} 5 & -4 \\ 11 & 9 \end{bmatrix}\right)\right)_{[95, \ldots, 100]}$$

$$\#48: \quad \begin{bmatrix} 6.4 & 2 \\ 6.4 & 2 \\ 6.4 & 2 \\ 6.4 & 2 \\ 6.4 & 2 \\ 6.4 & 2 \end{bmatrix}$$

I introduce a more general function including a variable (matrix) for the initial points:

Initial points are [-3, 5], [5, –4] and [11, 9]. What is the convergence point now, if there is one?

$$\#53: \quad \left(\mathrm{ptsG}\left(100, \begin{bmatrix} -3 & 5 \\ 5 & -4 \\ 11 & 9 \end{bmatrix}\right)\right)_{100}$$

#54: $\left[\frac{2040693588115225}{351843720888832}, \frac{1688849860263931}{562949953421312}\right]$

#55: [5.8, 3]

Can you find out the rule?

#60: $[0.4\cdot x_3 + 0.4\cdot x_2 + 0.2\cdot x_1,\ 0.4\cdot y_3 + 0.4\cdot y_2 + 0.2\cdot y_1]$

Proof this!!

What happens if $m \neq 0.5$?

#65: $\left(\text{ptsG}\left(10000, \begin{bmatrix} -3 & 5 \\ 5 & -4 \\ 11 & 9 \end{bmatrix}, 0.49\right)\right)_{10000}$

#66: $[4.913712357\cdot 10^{-35}, 2.546159996\cdot 10^{-35}]$

#71: $\left(\text{ptsG}\left(10000, \begin{bmatrix} -3 & 5 \\ 5 & -4 \\ 11 & 9 \end{bmatrix}, 0.51\right)\right)_{10000}$

#72: $[1.473751792\cdot 10^{35}, 7.609403727\cdot 10^{34}]$

Conjectures?? Proof it!!

So What Was The Role Of The CAS?

❶ Although the DNL #22 Article said that it was finding a limit, it really only gave our intuition a "nudge."

❷ To really know that (0.4, 0.4) is the limit, a proof was required. The CAS can not construct a proof. There is no button to push.

❸ This is where a "partnership" develops. The student, and instructor, have to understand what it is that the CAS and other technologies can do to help with the reasoning process.

❹ Visualization is a powerful aid. Sometimes it takes the form of graphical displays, other times it may be just to generate a large number of terms or examples, or, as it this case it was to give a display that made certain patterns "stick out."

❺ As instructors, we need to "let a thousand flowers bloom", i.e. let our students try their own strategies and exercise the limits of the CAS and other technologies. Our role is to gently critique and offer guidance through suggestions. In this case, a real strategy did not emerge until it became clear that the denominators were powers of 2. Everything else emerged from this very simple observation.

Postlude

The original article was from DERIVE Newsletter #22. Rüdiger Baumann sent a short note for DERIVE Newsletter pointing to the fact that little generalization leads to Edward Sawada´s "Misguided Missile" contribution (also from DNL#22).

Here Is What Rüdiger Saw

Rüdiger recommended the ITERATES-procedure because the recursive procedure is too slow.

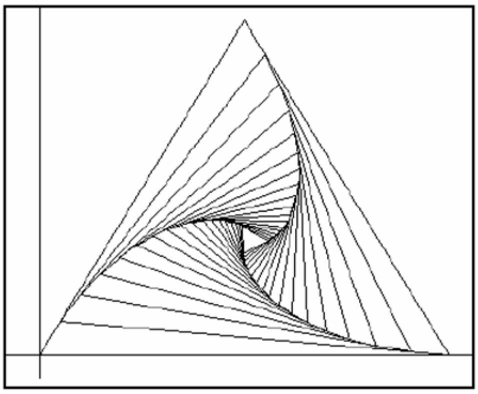
#85: pts_baum(r, s, ini, n) :: ITERATES([b, c, r·a + s·b], [a, b, c], ini, n)

This is the "Leinbach-Brubaker Sequence":

$$\#86: \quad \text{pts\_baum}\left(0.5,\ 0.5,\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},\ 50\right)$$

And this is Edward´s Missile:

$$\#87: \quad \text{pts\_baum}\left(0.9,\ 0.1,\ \begin{bmatrix} 0 & 0 \\ \dfrac{1}{2} & \dfrac{\sqrt{3}}{2} \\ 1 & 0 \end{bmatrix},\ 50\right)$$

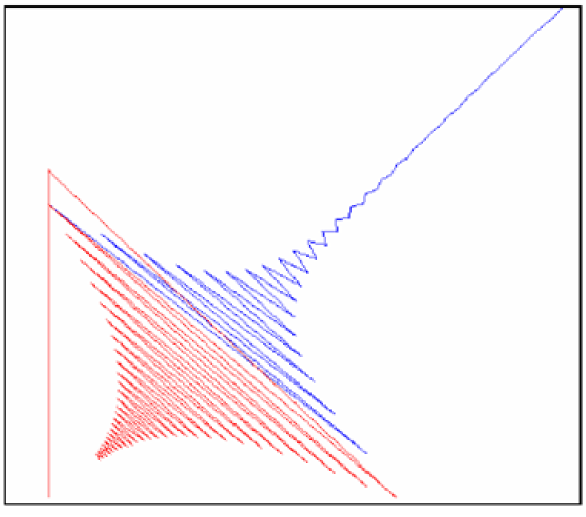


Experimenting with the parameters leads to interesting patterns. (Limits?)

$$\#89: \quad \text{pts\_baum}\left(0.15,\ 0.9,\ \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 2 & 0 \end{bmatrix},\ 50\right)$$

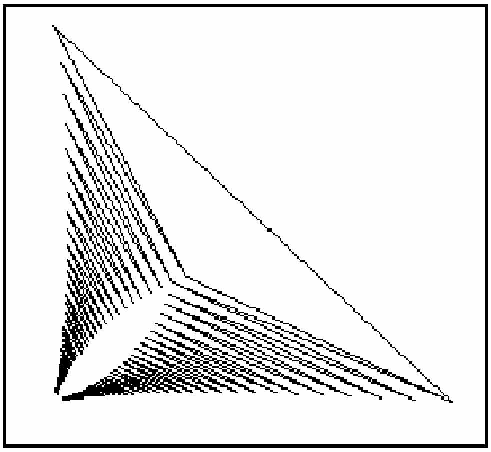
$$\#90: \quad \text{pts\_baum}\left(0.05,\ 0.9,\ \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 2 & 0 \end{bmatrix},\ 50\right)$$

#89 red and #90 blue

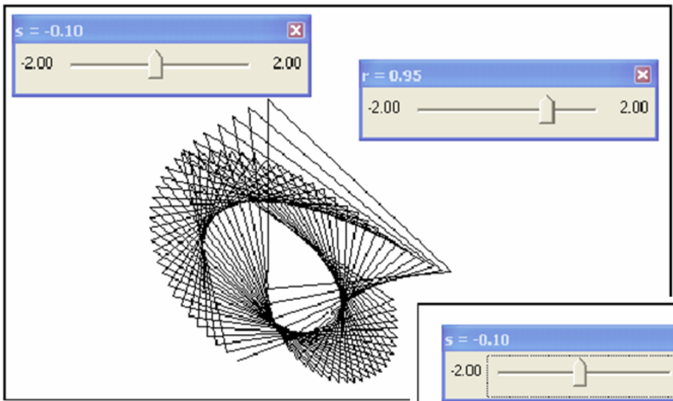


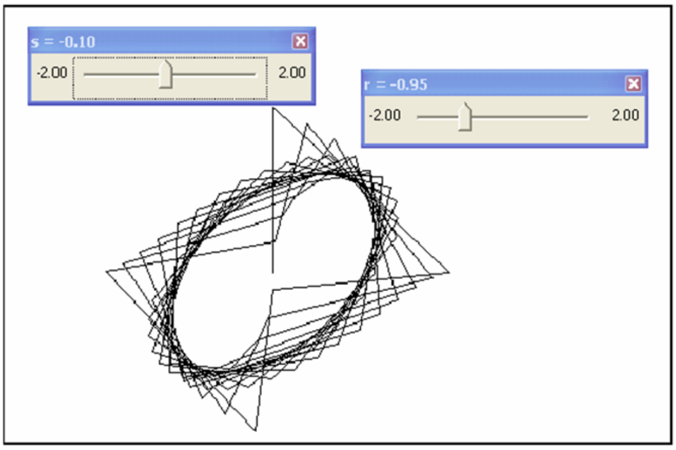
A Twin

$$\left[\text{pts\_baum}\left(0.02,\ 0.9,\ \begin{bmatrix} 0 & 3 \\ 1 & 1 \\ 3 & 0 \end{bmatrix},\ 80\right),\ \text{pts\_baum}\left(0.02,\ 0.9,\ \begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 1 & 1 \end{bmatrix},\ 80\right)\right]$$



Why not trying to introduce sliders for the parameters r and s and investigate their influence on the sequence of points?





Conics?

Proof this!

If you find another (better) proof, then please write to

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