Overcoming difficulties in understanding of the nonlinear programming concepts

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Annotation

This paper presents several examples of using computer technology in learning the nonlinear programming concepts.

Nonlinear optimization problems are often encountered in mathematical modeling of real processes.

The goal of studying is to help students overcome difficulties if they can not adequately formulate and solve the nonlinear optimization problems.

Contents

The traditional methods of teaching

The following questions are usually considered in the course «Operations Research and Mathematical Programming» for students (the Computer Science faculty at the Eastern-Ukrainian branch of the International Solomon University, Kharkov) during studying the theme "Nonlinear Programming":

- formulation of the nonlinear programming problems,
- close cooperation with the linear programming problems,
- geometric interpretation of the nonlinear programming problems,
- methods of solving the nonlinear programming problems.

Types of students’ errors

- Scope of feasible solutions was defined incorrectly.
- The objective function wasn’t named correctly.

Types of students’ errors

- Scope of permissible solutions was accurate, but the objective function was incorrect.
- The growth of the objective function was defined incorrectly.

Instructional unit

- A unit plan is a series of lessons organized around a single theme, topic, or mode. The unit plan should provide the teacher with a concise overview of the unit. The unit should be organized to emphasize sequences of learning activities.
- Goals and outcomes of a unit of instruction clarify what students should know and be able to do as a result of having instruction and learning through the unit's content and activities.

The “Nonlinear programming” unit

The additional content is the following:

- Formulation of the nonlinear optimization problems.
- Curves and surfaces and their classification.
- Contour lines and gradients.
- Contour lines as the intersection of a real or hypothetical surface with one or more horizontal planes.

Goals

The following questions were included in the course in order to overcome difficulties:

- consideration of the classic optimization problems;
- construction of the catalogue of surfaces (in Maple);
- some additional information about contour lines, contour surfaces, and gradient;
- the possibilities of the Maple package for geometric interpretation of the nonlinear programming problems.

The classic optimization problems

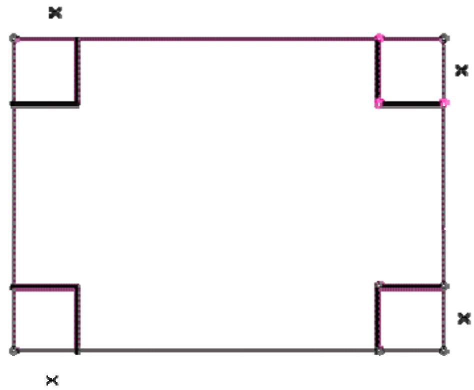
Problem 1. Consider a problem of dividing a given number ***A*** into two summands, so that their product was the highest.

- Ask the students which division would seem to yield a maximum value.
  - They guess that two numbers should be equal.
- Ask the students about mathematical formulation of the problem.
  - Formulation 1. Let ***x*** is one of the summands, ***A-x*** is a second summand. The value of the expression ***y = x (A-x)*** must be maximized.
- Solution. Calculate the derivative of ***y***.
  - They calculate the derivative ***y '= A-2x = 0*** and obtain ***x = A / 2***.

Simple problems

Problem 2. Consider a classic optimization problem of maximizing the volume of a box. The box's dimensions are 48 and 30.

The problem: illustrate the act of creating a box from the given rectangle by cutting equal squares from each corner and turning up.



Instructions

- Give each student a different side measure of X for the square to be cut off. Use X = 1…13 and 14. Discuss what happens if the side measure is equal to 15. This would be an appropriate time to discuss the domain of the X value.
- Ask the students which basic shape would seem to yield a maximum value. Should the box be tall, short, or cubical? Discuss each student's volume and which appears to have maximum volume.
- Ask "What about other values of " X" besides integers? Could x = 3.5? How can we look at all possible values of x and its corresponding volume?".

The mathematical model

- The equation determined for the volume of a box created by cutting out a square of area "x^2" should be ***V = x(18 - 2x)(24 - 2x).***
- Using this equation, we will now explore some methods of looking at all values of ***x*** that will produce a reasonable volume and how to locate these values through use of the spreadsheet and use of a graphing tool.

Using spreadsheet

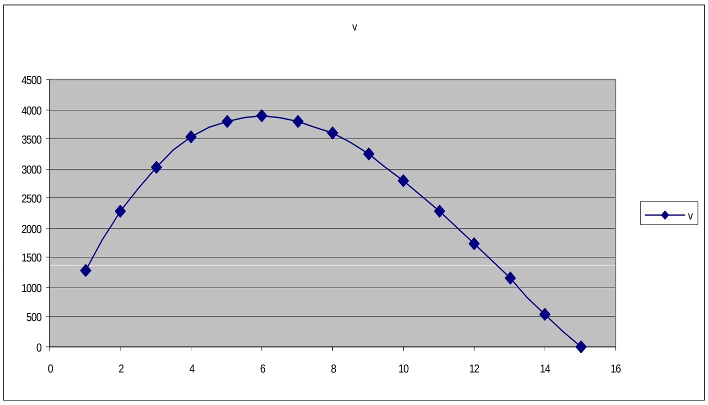
- We use Microsoft Excel to locate the length of the side of the square which gives the maximum volume.
- Students should begin by typing 1 into the A1 slot. To increase each time by an increment of 1, type `A1 + 1' into the A2 slot. Click on A2 and fill down to about row 16.
- Now go back up to B1 and type your volume formula discovered earlier using `A1' for `x'. Click on B1 and fill down to discover all the volumes.
- Find where the volume is the largest and its corresponding x value will tell you the dimensions of the square to remove from each corner to create maximum volume.

Using spreadsheet

The volume V is the largest if ***x***=6.

Graph the equation you discovered for the volume of the box.

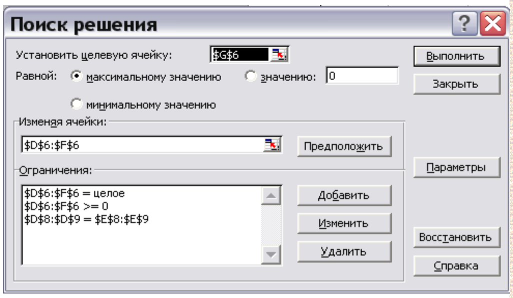
x	*v*
1	1288
2	2288
3	3024
4	3520
5	3800
6	**3888**
7	3808
8	3584
9	3240
10	2800
11	2288
12	1728
13	1144
14	560
15	0
max=	3888



Using spreadsheet

- How would you determine the dimensions of a box that has a definite volume?
  - We should use Goal Seek.
- How would you calculate ***x*** if the box has arbitrary dimensions?
  - We should use Solver.

x	*y*	*z*	*v*
6	36	18	**3888**
48	48		
30	30		



The mathematical models (problems 1-2)

Problem 1 Let x is one of the summands, and y is a second summand.

The mathematical formulation of the problem: to find the maximum of the objective function $z = xy$ under the constraint $x + y = A$ or

maximize

$$z = xy$$

subject to

$$x + y = A$$

Problem 2 Let x, y, z are the dimensions of the box.

Maximize

$$u=xyz$$

subject to

$$y = 48 - 2x, z = 30 - 2x$$

Mathematical programming problems

Mathematical programming problem is to maximize (minimize) the objective function under given constraints.

There are three main types of general problem of mathematical programming:

- a classical problem,
- the problem of nonlinear programming,
- linear programming problem.

NONLINEAR PROGRAMMING

$$\min_{x \in X} f(x),$$

where

- $f : \Re^n \mapsto \Re$ is a continuous (and usually differentiable) function of n variables
- $X = \Re^n$ or X is a subset of $\Re^n$ with a "continuous" character.

- If $X = \Re^n$, the problem is called unconstrained
- If f is linear and X is polyhedral, the problem is a linear programming problem. Otherwise it is a nonlinear programming problem
- Linear and nonlinear programming have traditionally been treated separately. Their methodologies have gradually come closer.

The examples of problems

maximize $z = 0{,}5x_1 + 2x_2$ subject to $\begin{cases} x_1 + x_2 \le 6 \\ x_1 - x_2 \le 1 \\ 2x_1 + x_2 \ge 6 \\ 0{,}5x_1 - x_2 \ge -4 \end{cases}$ $x_1 \ge 0, x_2 \ge 0$	minimize $z = 10(x_1 - 3{,}5)^2 + 20(x_2 - 4)^2$ subject to $\begin{cases} x_1 + x_2 \le 6 \\ x_1 - x_2 \le 1 \\ 2x_1 + x_2 \ge 6 \\ 0{,}5x_1 - x_2 \ge -4 \end{cases}$ $x_1 > 0, x_2 > 0$
maximize $z = -x_1^2 - x_2^2$ subject to $(x_1 - 7)^2 + (x_2 - 7)^2 \le 18, x_1 \ge 0, x_2 \ge 0$	minimize $z = x_1 + x_2$ subject to $x_1^2 + x_2^2 = 2$

A contour line and gradient

- A contour line for a function of two variables is a curve connecting points where the function has the same particular value.
- The gradient of the function is always perpendicular to the contour lines.
- When the lines are close together the magnitude of the gradient is large: the variation is steep.
- A level set is a generalization of a contour line for functions of any number of variables.

Level curves and level surfaces

- Level curves

If $f(x, y)$ is a function of two variables,
then $f(x, y) = c = const$ is a curve or a collection of curves in the plane. It is called contour curve or level curve.

For example,
$f(x, y) = 4x^2 + 3y^2 = 1$ is an ellipse.

- Level surfaces

There is 3D analogue:
if $f(x, y, z)$ is a function of three variables and c is a constant then $f(x, y, z) = c$ is a surface in space.

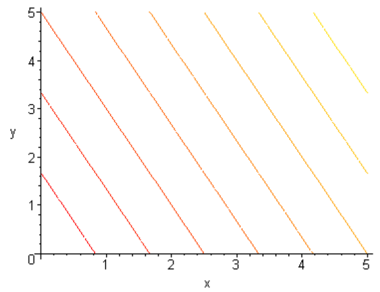
It is called a contour surface or a level surface.

For example if $f(x, y, z) = 4x^2 + 3y^2 + z^2$ then the contour surfaces are ellipsoids.

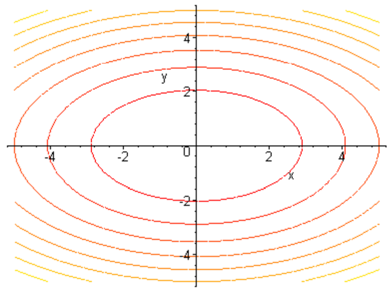
Level curves

Level curves allow to visualize the objective functions of two variables ***f(x, y)***.

- If the objective function ***f(x, y) = 2x+y***, then the level curves ***f(x, y) = 2x+y = c*** are the straight lines.
- If the objective function ***f(x, y)*** = ***$2x^2 + y^2$*** , then the level curves ***f(x, y)*** = ***$2x^2 + y^2 = c$*** are the ellipses.



The straight lines

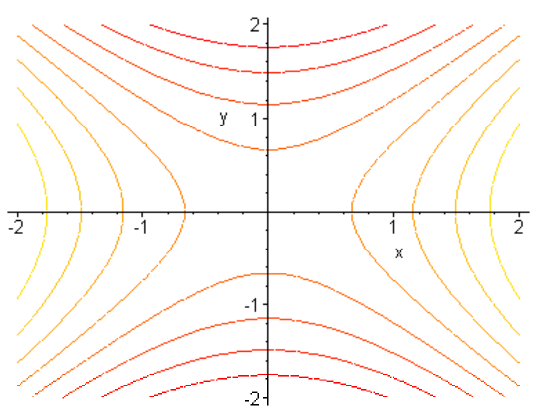


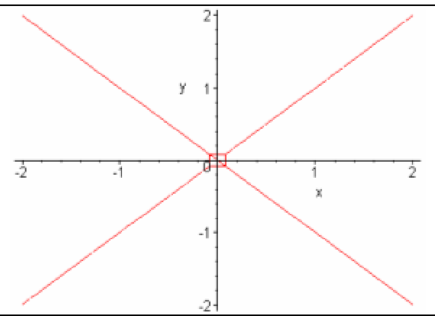
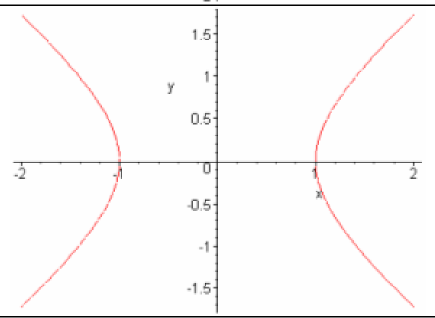
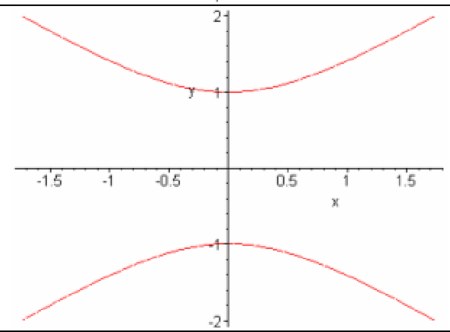
The ellipses

Level curves

If the objective function ***f(x, y)*** = ***$x^2 - y^2$*** then the level curves ***f(x, y)*** = ***$x^2 - y^2 = c$*** are the hyperbola.

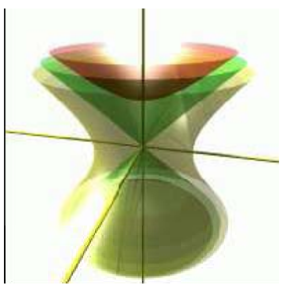
➢ *contourplot(x^2-y^2,x=-2..2,y=-2..2);*



The curve $x^2 - y^2 = 0$ is the union of $x = y$ and $x = -y$. **implicitplot**(x^2-y^2=**0**,x=-2..2,y=-2..2);	
The curve $x^2 - y^2 = 1$ consists of two hyperbola with their vertices at (-1; 0) and (1; 0). **implicitplot**(x^2-y^2=**1**,x=-2..2,y=-2..2);	
The curve $x^2 - y^2 = -1$ consists of two hyperbola with their vertices at (0; 1) and (0; -1). **implicitplot**(x^2-y^2=**-1**,x=-2..2,y=-2..2);	

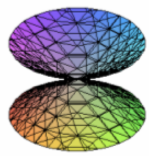
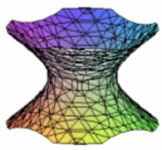
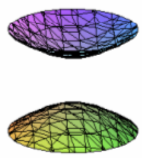
Level surfaces

If the objective function $f(x, y, z) = x^2 + y^2 - z^2$,
then the level surfaces $f(x, y) = x^2 - y^2 - z^2 = c$ are
the hyperboloids



The hyperboloid

Catalogue of surfaces

The surface $x^2 + y^2 - z^2 = 0$ is a cone rotational symmetric around the z-axis.	
The surface $x^2 + y^2 - z^2 = 1$ is a one-sheeted hyperboloid.	
The surface $x^2 + y^2 - z^2 = -1$ is a two-sheeted hyperboloid.	

Tasks for students

- Name the type of the problem
- Name the type of the objective function
- Name the type of the constraints
- Define the set of constraints in Maple
- Define the level curve in Maple
- Investigate the direction of growth for the objective function in Maple
- Explain the result
- Experiment with the constraints
- Experiment with the objective function

Using Maple

Example 1. The linear programming problem:

maximize $z = 0{,}5x_1 + 2x_2$

subject to $\begin{cases} x_1 + x_2 \le 6 \\ x_1 - x_2 \le 1 \\ 2x_1 + x_2 \ge 6 \\ 0{,}5x_1 - x_2 \ge -4 \end{cases}$

$x_1 \ge 0, x_2 \ge 0$

The scope of permissible solutions is shown in the figure; it is the convex polygon ABCD

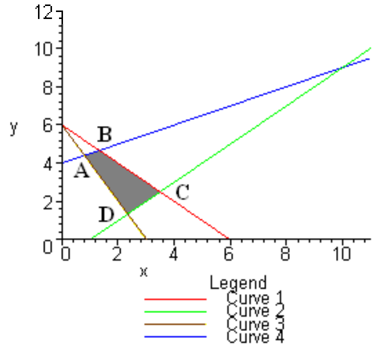
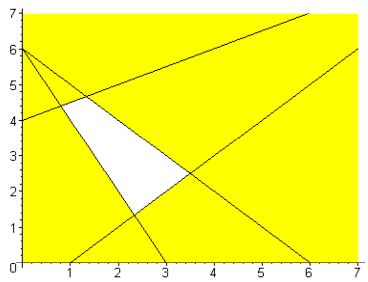


Fig. 1. The scope of permissible solutions

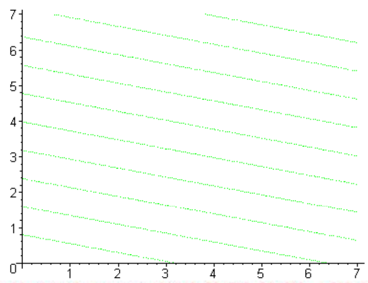

```
> CS:=[x[1]+x[2]<=6,x[1]-x[2]<=1,2*x[1]+x[2]>=6,0.5*x[1]-x[2]>=-4,x[1]>=0,x[2]>=0];
```

$CS := [x_1 + x_2 \le 6, x_1 - x_2 \le 1, 6 \le 2\,x_1 + x_2, -4 \le .5\,x_1 - x_2, 0 \le x_1, 0 \le x_2]$

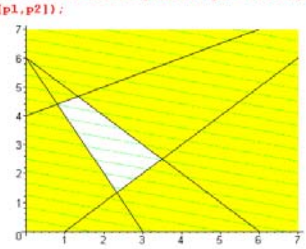
```
> inequal(CS, x[1]=0..7, x[2]=0..7,optionsfeasible=(color=white),optionsexcluded=(color=yellow));
```



```
> contourplot(0.5*x[1]+2*x[2], x[1]=0..7, x[2]=0..7,contours=10,linestyle=2,numpoints=2000,color=green);
```



```
> p1:=inequal(CS, x[1]=0..7, x[2]=0..7,optionsfeasible=(color=white),optionsexcluded=(color=yellow)):p2:=contourplot(0.5*x[1]+2*x[2], x[1]=0..7,
x[2]=0..7,contours=20,linestyle=2,numpoints=2000,color=green): display([p1,p2]);
```



```
>
```

Finding the vertices in Maple

```
> CS:=[x[1]+x[2]<=6,x[1]-x[2]<=1,2*x[1]+x[2]>=6,x[1]/2-x[2]>=-4,x[1]>=0,x[2]>=0];
```

$$CS := \left[x_1 + x_2 \le 6, x_1 - x_2 \le 1, 6 \le 2x_1 + x_2, -4 \le \frac{1}{2}x_1 - x_2, 0 \le x_1, 0 \le x_2\right]$$

```
> with(simplex);
> solve(convert({CS[4],CS[3]},equality)); assign(%); z:=0.5*x[1]+2*x[2]; x[1]:='x[1]': x[2]:='x[2]':
```

$$\{x_2 = \frac{22}{5}, x_1 = \frac{4}{5}\}$$

$$z := 9.200000000$$

```
> solve(convert({CS[4],CS[1]},equality)); assign(%); z:=0.5*x[1]+2*x[2]; x[1]:='x[1]': x[2]:='x[2]':
```

$$\{x_1 = \frac{4}{3}, x_2 = \frac{14}{3}\}$$

$$z := 10.00000000$$

```
> solve(convert({CS[1],CS[2]},equality)); assign(%); z:=0.5*x[1]+2*x[2]; x[1]:='x[1]': x[2]:='x[2]':
```

$$\{x_1 = \frac{7}{2}, x_2 = \frac{5}{2}\}$$

$$z := 6.750000000$$

```
> solve(convert({CS[2],CS[3]},equality)); assign(%); z:=0.5*x[1]+2*x[2]; x[1]:='x[1]': x[2]:='x[2]':
```

$$\{x_1 = \frac{7}{3}, x_2 = \frac{4}{3}\}$$

$$z := 3.833333334$$

We found the vertices:

A(4/5,22/5), B(4/3,14,3), C(5/2,7/2), D(7/3,4/3)

Example 1

The maximum is in the point B(4/3, 14/3).

```
> p1:=plot ([6-x, -1 + x ,6-2 * x, 0.5 * x +4], x = 0 .. 7, y = 0 .. 7,color=black,
  thickness=2):p2:=contourplot (0.5 * x +2 * y, x = 0 .. 7, y = 0 .. 7, contours =
  [92/10,10,135/20,23/6],linestyle=2,numpoints=2000,color=red):p3:=gradplot(
  0.5 * x +2 * y, x = 0 .. 7, y = 0 .. 7,linestyle=3,color=navy):display([p1,p2,p3]);
```

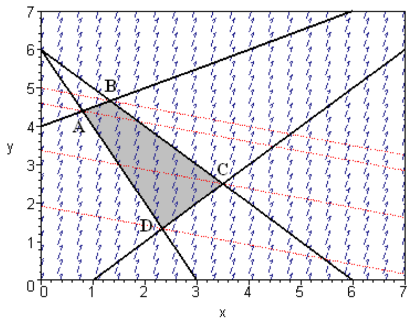


Fig. 2. The level lines

Outcome. Example 1

This example illustrates one of possible solutions of linear programming problem.

- The linear objective function gives rise to the level of straight lines.
- The linear constraints form a feasible set bounded by line segments.
- Since the objective function is linear, then the direction in which the objective function increases with the maximum speed, is the same everywhere.
- In this case, the solution may be found either at the vertex or on the linear segment.
- The objective function takes the maximum value at a boundary point of the region - at the point B(4/3,14/3).

Example 2

Example 2. Consider the situation when the objective function is nonlinear, and the restrictions are the same.

Find the minimum of the nonlinear objective function

$$z = 10(x_1 - 3{,}5)^2 + 20(x_2 - 4)^2$$

under linear constraints

$$\begin{cases} x_1 + x_2 \le 6 \\ x_1 - x_2 \le 1 \\ 2x_1 + x_2 \ge 6 \\ 0{,}5x_1 - x_2 \ge -4 \end{cases}$$

$$x_1 \ge 0, x_2 \ge 0$$

Scope of feasible solutions, together with level lines, can be constructed using the Maple operator:

```
> p3:= plot ([6-x, -1 + x ,6-2 * x, 0.5 * x +4], x = 0 .. 7, y = 0 .. 7): p4:=
  contourplot (10 * (x-3.5) ^ 2 +20 * (y-4) ^ 2, x = 0 .. 7, y = 0 .. 7,
  contours = [15,76,56,45,156],linestyle=1,color=black):p5:=gradplot(10
  * (x-3.5) ^ 2 +20 * (y-4) ^ 2,x=0..7,y=0..7): display ([p3, p4,p5]);
```

Example 2

- The objective function is a family of ellipses centered at the point $x_1 = 3{,}5; x_2 = 4$
- The optimal solution is in the point M, at which the ellipse has the first touch the convex polygon ABCD.

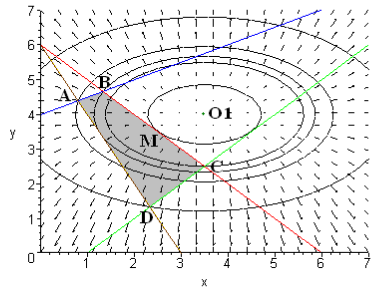
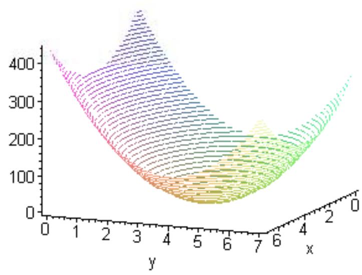


Fig. 3. The level curves for the problem of Example 2

Contour surface

- > contourplot3d
  (10 * (x-3.5) ^ 2 +20 * (y-4) ^ 2,
  x = 0 .. 7, y = 0 .. 7, contours =
  25);



Example 3

Example 3. Let us consider another objective function:

$$z = 10(x_1 - 2)^2 + 10(x_2 - 3)^2$$

The level curves of this function is a family of circles centered at the point M (2, 3)

The minimum of the objective function Z is in the point M.

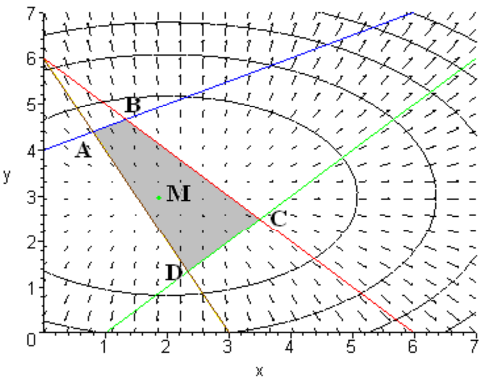


Fig. 4. The level curves for the problem of Example 3

Example 4

Example 4. Find the maximum of the nonlinear function

$$z = -x_1{}^2 - x_2{}^2$$

with nonlinear constraints:

$$(x_1 - 7)^2 + (x_2 - 7)^2 \le 18, x_1 \ge 0, x_2 \ge 0$$

The problem can be interpreted using the operator Maple:

```
> p1: = contourplot (-x ^ 2-y ^ 2, x = 0 .. 10, y = 0 .. 10, filled = true,
contours = 30): p2: = plot ([7 + sqrt (18 - (x-7) ^ 2) ,7-sqrt (18 - (x-7) ^ 2)],
x = 0 .. 10, y = 0 .. 10, color = [green, green]): p3: = plot ([[7,7]], style =
point, color = green): display ([p1, p2, p3]);
```

Example 4

If the objective function and constraint functions are nonlinear, then under appropriate assumptions on the convexity, the problem of classical programming has a unique solution in the osculation point, at which two branches of a curve have a common tangent, each branch extending in both directions of the tangent.

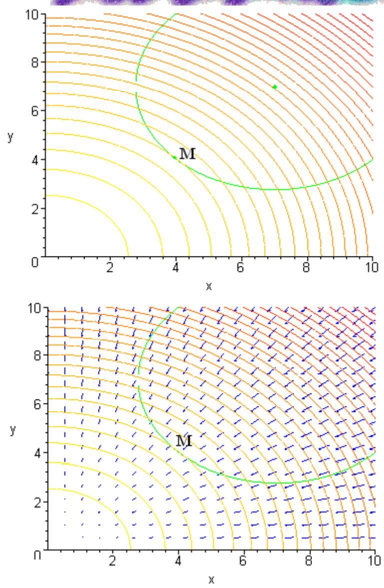


Fig. 5. The level curves and gradient field for the problem of Example 4

Using Maple (Problem 1)

- *pict1:=contourplot(x*y,x=0..10,y=0..10, contours=35): pict2:=implicitplot(10-x-y,x=0..10,y=0..10):*
display([pict1,pict2]);

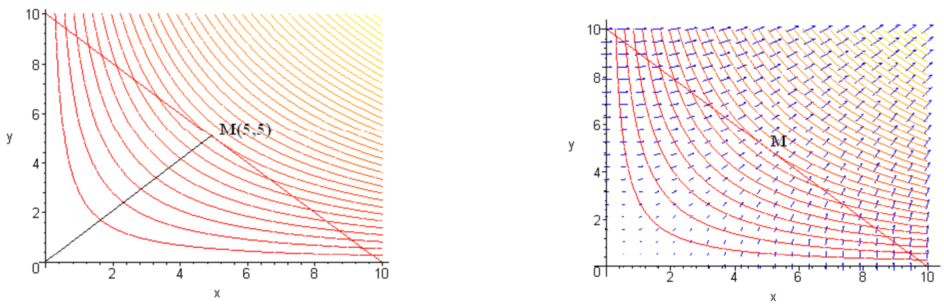


Fig. 5. Solution with A=10

CONSTRAINED OPTIMIZATION;

LAGRANGE MULTIPLIERS

Equality constrained problem

minimize $f(x)$

subject to $h_i(x) = 0, \qquad i = 1, \ldots, m.$

where $f : \Re^n \mapsto \Re$, $h_i : \Re^n \mapsto \Re$, $i = 1, \ldots, m$, are continuously differentiable functions. (Theory also applies to case where f and h_i are cont. differentiable in a neighborhood of a local minimum.)

LAGRANGE MULTIPLIERS

Let x^* be a local min and a regular point [$\nabla h_i(x^*)$: linearly independent]. Then there exist unique scalars $\lambda_1^*, \ldots, \lambda_m^*$ such that

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla h_i(x^*) = 0.$$

Using Lagrange multipliers method

1) We form the Lagrange function for our Problem1:

L(x, y, λ)=xy+λ(A-x-y)

From equations we find the stationary points:

- $\lambda = A / 2$
- $x = y = A / 2$

$$\begin{cases} \dfrac{\partial L}{\partial x} = y - \lambda = 0 \\ \dfrac{\partial L}{\partial y} = x - \lambda = 0 \\ \dfrac{\partial L}{\partial \lambda} = A - x - y = 0 \end{cases}$$

2) We form the Lagrange function for our Problem2:

L(x, y, z, λ1, λ2)=xyz+ λ1(48-2x-y)+ λ2(30-2x-z)

```
> L:=x*y*z+lambda1*(48-2*x-y)+lambda2*(30-2*x-z);
```

$$L := x\,y\,z + \lambda 1\,(48 - 2\,x - y) + \lambda 2\,(30 - 2\,x - z)$$

```
> t1:=diff(L,x)=0;
```

$$t1 := y\,z - 2\,\lambda 1 - 2\,\lambda 2 = 0$$

```
> t2:=diff(L,y)=0;
```

$$t2 := x\,z - \lambda 1 = 0$$

```
> t3:=diff(L,z)=0;
```

$$t3 := x\,y - \lambda 2 = 0$$

```
> t4:=diff(L,lambda1)=0;
```

$$t4 := 48 - 2\,x - y = 0$$

```
> t5:=diff(L,lambda2)=0;
```

$$t5 := 30 - 2\,x - z = 0$$

Using Maple (Problem 2)

```
> p:={t1,t2,t3,t4,t5};
```

$$p := \{y\,z - 2\,\lambda 1 - 2\,\lambda 2 = 0,\ x\,z - \lambda 1 = 0,\ x\,y - \lambda 2 = 0,\ 48 - 2\,x - y = 0,\ 30 - 2\,x - z = 0\}$$

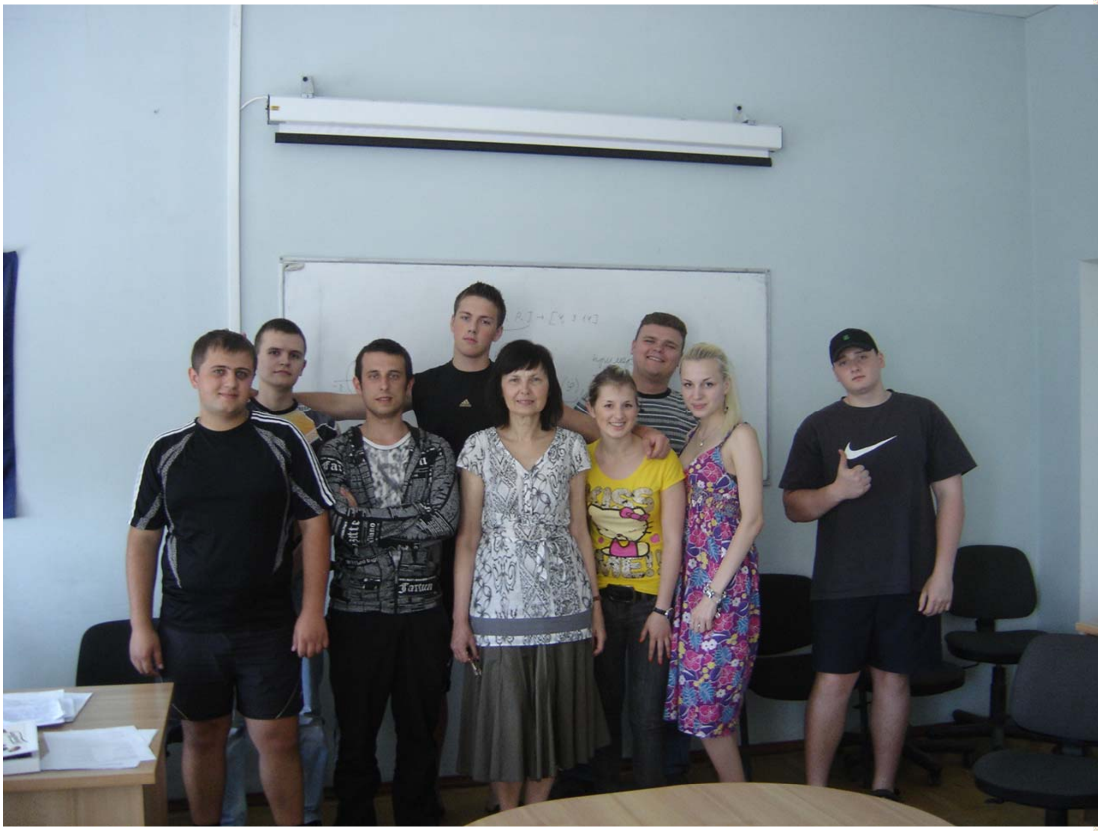
```
> solve(p,{x,y,z,lambda1,lambda2});
```

$$\{x = 6,\ \lambda 1 = 108,\ \lambda 2 = 216,\ y = 36,\ z = 18\},$$
$$\{x = 20,\ \lambda 2 = 160,\ \lambda 1 = -200,\ y = 8,\ z = -10\}$$

As ***x*** <15 for Problem 2 we choose the first solution.

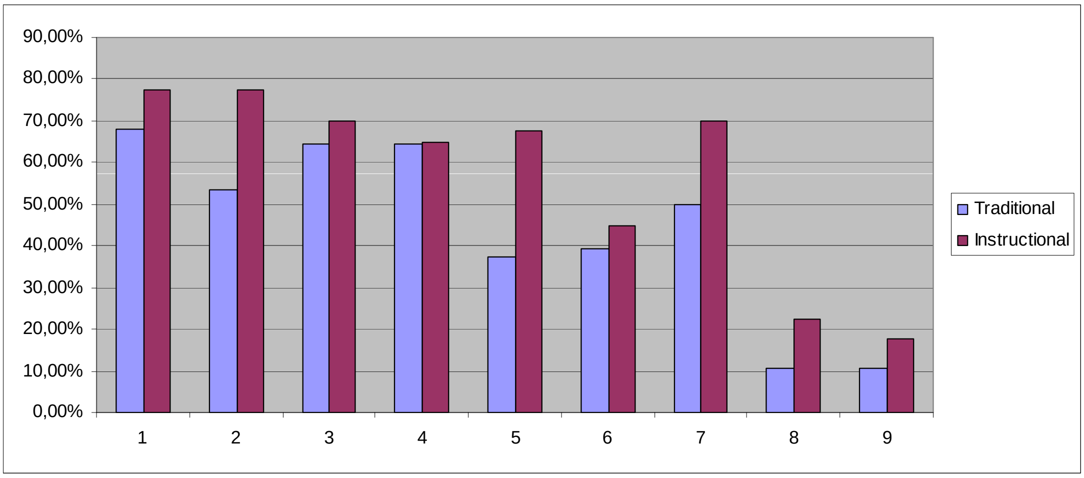
x=6, y=36, z=18, λ1=108, λ2=216



Outcome

	Question	Traditional	Instructional (with Maple)
1	Name the type of the problem	67,86%	77,50%
2	Name the type of the objective function	53,57%	77,50%
3	Name the type of the constraints	64,29%	70,00%
4	Define the set of constraints	64,29%	65,00%
5	Define the level curve	37,50%	67,50%
6	Investigate the direction of growth for the objective function	39,29%	45,00%
7	Explain the result	50,00%	70,00%
8	Experiment with constraints	10,71%	22,50%
9	Experiment with the objective function	10,71%	17,50%



Conclusions

- The active using the graphic representation has enabled to overcome the difficulties while studying the basic concepts of nonlinear programming.
- It has resulted in higher performance and longer information retention compared to traditional methods of teaching.

Thank You for attention!