

# Folding and Unfolding a Cone with Cabri 3D : a Process of Modelling Generating New Problems to Be Explored

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**Abstract:** As Cabri 3D does not contain the tool “net of a cone”, a middle school teacher complains to me because she needed it to help her students to understand that the net of a cone is not always half a disk. I have explored this problem and present a process of modelling the (un)folding of a cone. This result allows to open a window on new problems that can be explored: some invariants about the net of a cone and some 3D curves obtained when unfolding the cone. By the way we enrich the cooperation between a researcher and a teacher. Some of the problems explored thanks to my model can be posed to the students of the middle school teacher and enhance their experimental practice of mathematics using ICT

## 1. Introduction

### 1.1. First approach

As a circle can be approximated with a sequence of regular polygons inscribed in this circle, a cone can be approximated with a sequence of regular pyramids inscribed in this cone. As Cabri 3D returns the net of any polyhedron and as this net can be folded or unfolded, this net could have been a good tool to provide to the middle school teacher a response to her pedagogical problem. Helas, the net provided by Cabri 3D for such a pyramid is not the one expected because it does not unfold around the summit of the pyramid (figure 1 on the right). The one expected, obtained after cutting the cone along a generatrix would have been like in figure 1 on the left.

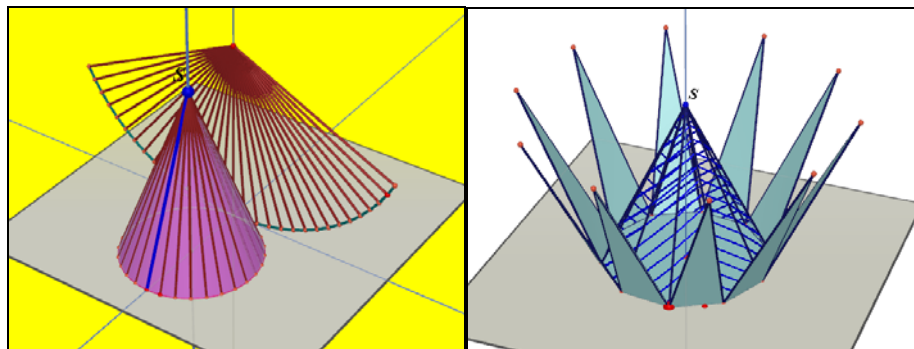


Figure 1

### 1.2. Second approach

We still want to use a regular pyramid as a model for a cone. This model will be all the more accurate than the number of its lateral faces is big. Let us choose  $2n$  as the number of the lateral faces (which are isosceles triangles) of the pyramid we use for modelling a cone. We will use now transformations in order to open our pyramid like two wings from its closed initial position to the very last one which will be the opened and flat position.

Crucial remark: it is possible to obtain all the faces of the  $2n$ -pyramid in knowing only the first one and in using reflections with respect to planes we will specify. So we can get an algorithm of construction of the unfold wings modelling the unfolding of the pyramid.

Description of this algorithm (figure 2): in this figure  $B_1, B_2, \dots, B_{n+1}, B'_n, \dots, B'_2, B_1$  is the sequence of points defining a regular  $2n$ -polygon inscribed in the circle which is the base of our initial cone. Given only points  $B_1, B_2$  and  $S$ , that is to say the initial isosceles triangle defining our regular  $2n$ -pyramid ( $S$  is its summit), here is the process of construction of this pyramid that will be used later:

Construct triangle  $B_1B'_2S$  as the symmetric of triangle  $B_1B_2S$  with respect to the  $xB_1z$  plane

Construct triangle  $B_2B_3S$  as the symmetric of triangle  $B_1B'_2S$  with respect to the perpendicular bisector plane between  $B_1$  and  $B_2$ .

If triangle  $B_{k-1}B_kS$  is constructed, the next one  $B_kB_{k+1}S$  is obtained as the symmetric of the triangle  $B_{k-2}B_{k-1}S$  with respect to the perpendicular bisector plane between  $B_{k-1}$  and  $B_k$ .

When all triangles between  $B_1B_2S$  and  $B_nB_{n+1}S$  are constructed, their symmetric with respect to the  $xB_1z$  plane will give all the missing triangles between  $B_1B'_2S$  and  $B'_nB'_{n+1}S$  ( $B_{n+1} = B'_{n+1}$ ).

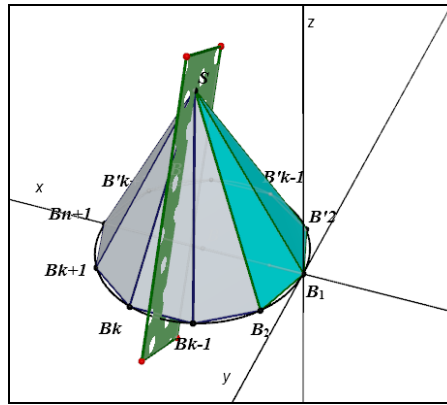


Figure 2

## 2. Un(folding) a $2n$ -regular pyramid

### 2.1. Onto a tangent plane of the initial cone

-The cone is cut along  $SB_{n+1}$ . The plane on which we will unfold the cone (really the pyramid modelling this cone) is the plane defined by  $(SB_1)$  and the  $y$  axis (figure 3 on the left). In order to use the previous algorithm, let us construct the first face  $SB_1B_2$  of our  $2n$ -pyramid modelling our cone. Let us display an integer for  $n$  and the value of  $360^\circ/n$ .  $B_2$  is the image of point  $B_1$  with the rotation around  $(SB)$  having the displayed number  $360^\circ/n$  as an angle (figure 3 on the middle). As the value of  $n$  can be changed at any moment, the angle of the rotation can be decreased as much as we want.

-The trick of this construction is the choice of the way to “unfold” the triangle  $SB_1B_2$  from its initial position to a position on the tangent plane. So we create a circle having  $SB_1$  as an axis and passing through  $B_2$ . This circle cuts the tangent plane on  $C_2$ . On this circle we create the arc  $\overset{\frown}{B_2C_2}$  and a point  $M_2$  on this arc (figure 3 on the middle). The triangle  $SB_1M_2$  is the initial triangle on which we will apply our previous algorithm to obtain the unfolded  $2n$ -pyramid that can be dragged in dragging point  $M_2$  from  $B_2$  to  $C_2$  (figure 3 on the right).

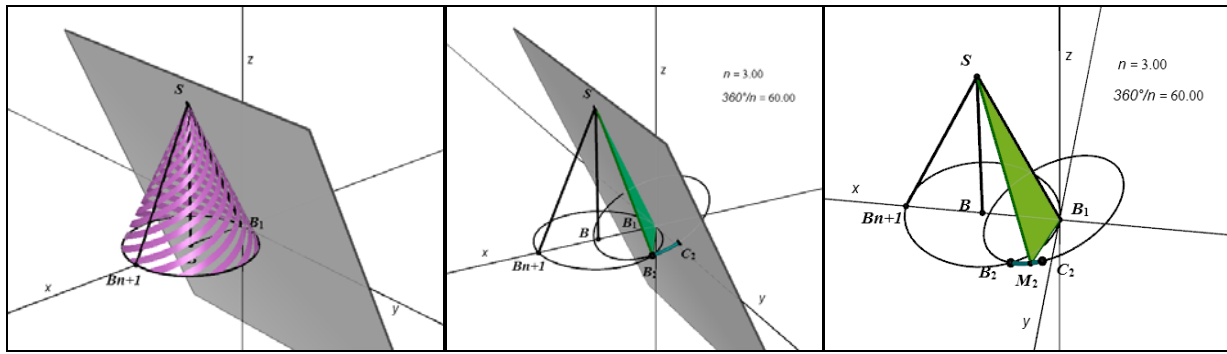


Figure3

The chosen value of  $n$  in figure 3 ( $n = 3$ ) is not big enough to obtain an acceptable model, but it will help us to realise the constructions given by the algorithm.

-Another technical problem needs to be solved: when we will increase the value of  $n$  to 12 to get a net of a 24-regular pyramid, it will be impossible to grab point  $M_2$  in order to drag it along the arc  $\overline{B_2C_2}$ . The trick is to create a slider, the segment  $[S_1S_2]$ : the motion of a point  $P$  along this segment will command the motion of point  $M_2$  along this tiny arc with a great accuracy. The technical work on Cabri 3D is quite simple: calculate and display  $\text{length}(\overline{B_2C_2}) * S_1P/S_1S_2$ . Transfer this measurement on the last circle constructed from point  $B_2$ : choose the orientation of this transfer in order to obtain a point belonging to the arc  $\overline{B_2C_2}$ ; at last redefine point  $M_2$  to be this point. Therefore, dragging point  $P$  along  $[S_1S_2]$  from  $S_1$  to  $S_2$  will generate a continuous motion of point  $M_2$  on arc  $\overline{B_2C_2}$  from  $B_2$  to  $C_2$ .

-Applying the algorithm to triangle  $SB_1M_2$ :

**First** (figure 4 on the left), construct triangle  $SB_1M'_2$  as the symmetric of triangle  $SB_1M_2$  with respect to the  $xB_1z$  plane; hide the maximum of objects in order to facilitate the next stages of the algorithm.

**Second** (figure 4 on the middle), construct triangle  $SM_2M_3$  as the symmetric of triangle  $SB_1M'_2$  with respect to the perpendicular bisector of points  $B_1$  and  $M_2$ . Iterate this process in order to obtain the sequence  $B_1, M_1, M_2, M_3, \dots, M_{n+1}$ . Stop when you have got  $M_{13}$  even if the value displayed for  $n$  is not 12 (we will change this value later).

**Third** (figure 4 on the right), construct the symmetric triangles of these ones with respect to the  $xB_1z$  plane. We have got a model of a folding net commanded by point  $P$ . Figure 3 does not display all the 24 triangles in order to be more readable and so, more understandable.

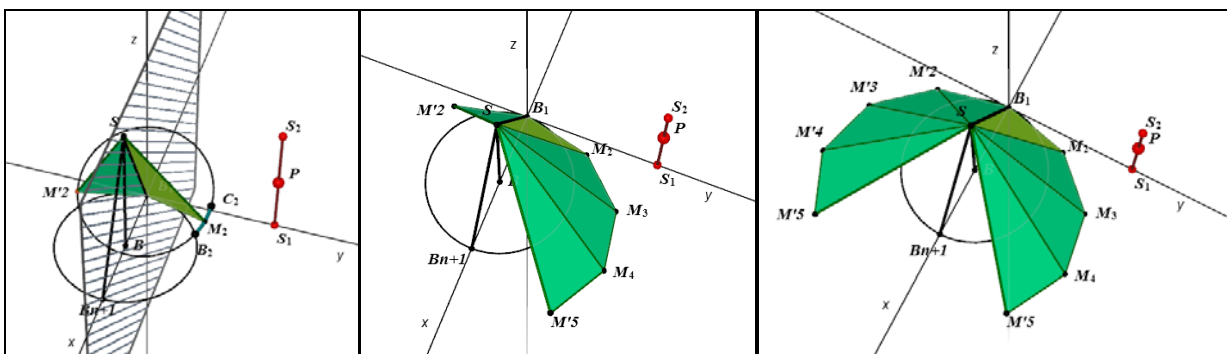
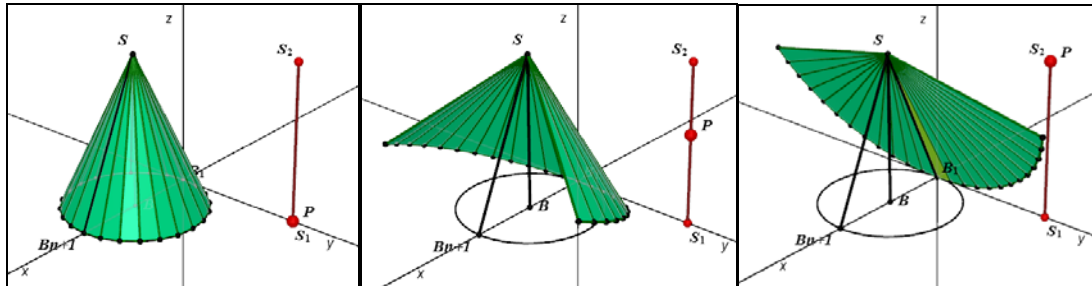


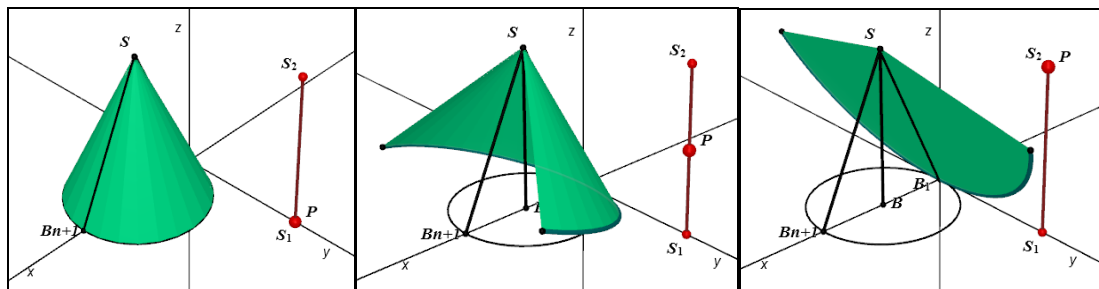
Figure 4

**At last:** after having constructed the 24 triangles, let us change the value of  $n$  onto 12; commanded by point  $P$ , we have realised for our middle school teacher a model of the net of a cone that can be folded and unfolded (figure 5 on the left and on the middle). The unfolded one will belong to the tangent plane to the cone along  $SB_1$  (figure 5 on the right). This will be improved in the next paragraph in order to obtain the unfolded net on the vertical  $yB_1z$  plane.



**Figure 5**

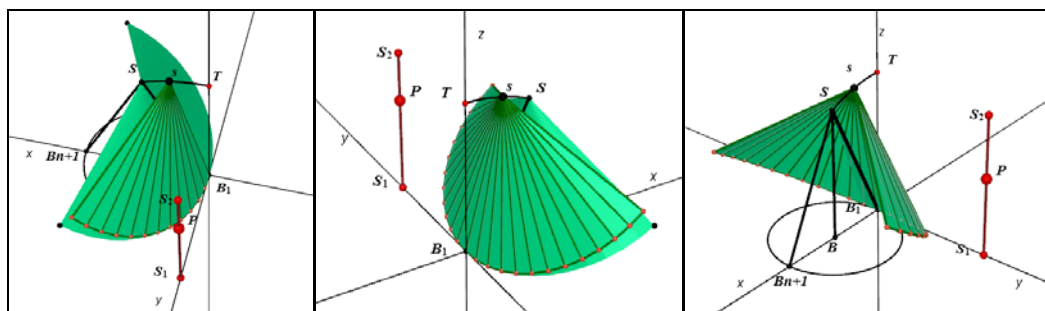
**A last improvement in the design** (figure 6) of this file: until now, we see the net of a pyramid and not the net of a cone. To hide this default, we choose an empty border style and an empty point style for each constructed triangle. We can also delete the option “display object hidden parts” in the Window options, so the objects will not be transparent.



**Figure 6**

## 2.2. Onto a parallel plane to the vertical axis of the cone

Let us create a circle around the  $y$  axis passing through  $S$ . It cuts the upper part of the  $z$  axis in  $T$ . Let us create an arc on this circle from  $S$  to  $T$  and a point  $s$  on this arc. We transform now all the triangles we have constructed with the rotation around the  $y$  axis mapping point  $S$  towards to  $s$  (figure 7 on the left and on the middle). To finish, we hide the model obtained in the previous paragraph (figure 7 on the right).



**Figure 7**

The border style of all triangles can be changed like we did before. But we can add the border of the cone (figure 6 on the left). The summit  $s$  of our new model can be dragged from  $S$  to  $T$  and so the net can be unfolded on the  $yB_1z$  plane when  $P$  reaches  $S_2$  (figure 8 on the middle). To return to the initial and given cone,  $s$  must be on  $S$  and  $P$  must be on  $S_1$  (figure 8 on the right).

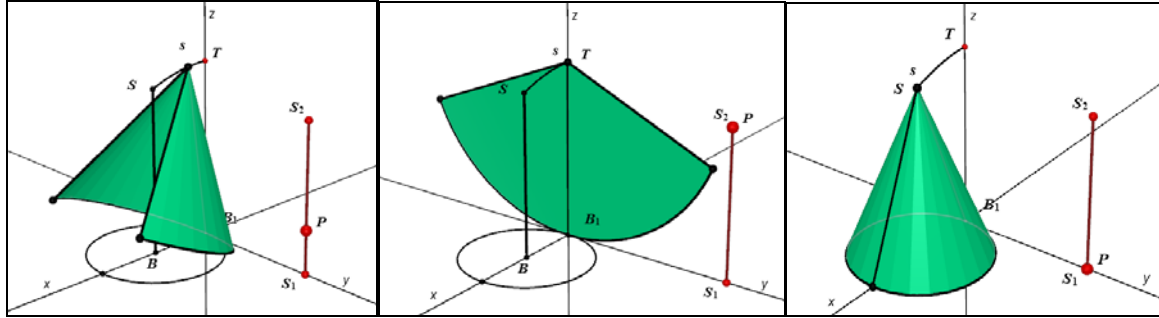


Figure 8

### 3. Some explorations with this model

#### 3.1. When the unfolded cone is half a disk (figure 9 on the left)

Experiments and conjecture: we display the measurements in degrees of  $\angle ESB_1$  and  $\angle F'TF$ ; we change the parameters of the initial cone in dragging either  $B$  to enlarge the radius of the basis circle or  $S$  to change the height of the cone until  $\angle F'TF$  reaches  $180^\circ$ : we observe that when this case occurs,  $\angle ESB_1$  is displayed as  $60^\circ$ . For the other positions of  $B$  and  $S$ , we get the same observation. So we can plausibly conjecture that the only cones that can be unfolded onto half a disk are cones such as the top angle is  $60^\circ$ .

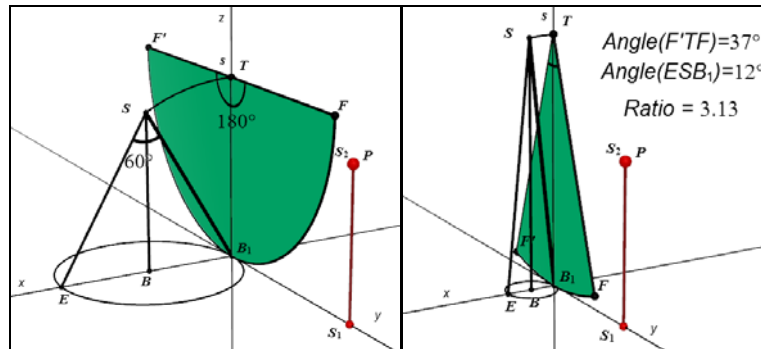


Figure 9

Formal proof (figure 9 on the left): if  $SB = h$  and  $BB_1 = r$ , therefore  $SB_1 = \sqrt{r^2 + h^2}$ ; having a cone unfolding in half a disk means that  $\angle F'TF = \pi$  rad or  $\frac{2\pi r}{\sqrt{r^2 + h^2}} = \pi$  which is equivalent to  $\frac{h}{r} = \sqrt{3}$  and to  $\angle BSB_1 = 30^\circ$  or  $\angle ESB_1 = 60^\circ$ . That is the end of this proof. This proof that can be discovered by a french student at the end of the middle school after the experimental work leading to the conjecture if the model of the cone is available.

### 3.2. Playing with parameters can lead to discovery (figure 9 on the right)

Experiments and conjecture: we know that very often in geometry as well in algebra, a good way to explore a problem is to transform a constant into a variable. Let us change the position of B and S such as  $BB_1$  becomes very small (B approaches  $B_1$ ) and SB as big as possible (S is lifted up). Let us observe the values taken by the ratio  $\frac{\angle F'TF}{\angle ESB_1}$ . The values displayed increase and give the impression to approach  $\pi$ : we can obtain very easily a ratio of 3,1415 on a normal Cabri page. So we can plausibly conjecture that the limit of this ratio is  $\pi$  when  $\frac{h}{r}$  goes to infinity.

Formal proof: as  $\angle ESB_1 = 2\text{Arc tan} \frac{r}{h}$ ,  $\lim_{\frac{r}{h} \rightarrow 0} \frac{\angle F'TF}{\angle ESB_1} = \lim_{\frac{r}{h} \rightarrow 0} \frac{\frac{2\pi r}{\sqrt{r^2 + h^2}}}{2\text{Arc tan}(\frac{r}{h})} \lim_{\frac{r}{h} \rightarrow 0} \frac{\frac{2\pi r}{h}}{2\frac{r}{h}} = \pi$ . At last, we

have proven that the ration between the angle of the unfolded cone and the top angle of the cone admits  $\pi$  as a limit when  $\frac{h}{r}$  goes to infinity or when  $\frac{r}{h}$  goes to 0.

### 3.3. Visualizing a property of the basis of the unfolded cone (figure 10)

If we create an arc from  $F'$  to  $F$  passing through  $B_1$ , we can observe that this arc seems to be superimposed to the basis of the unfolded cone (figure 10 on the left). It is confirmed by the construction of the plane containing these three points: this plane seems always to contain this basis (figure 10 on the right). The proof is easy (properties of the reflections)

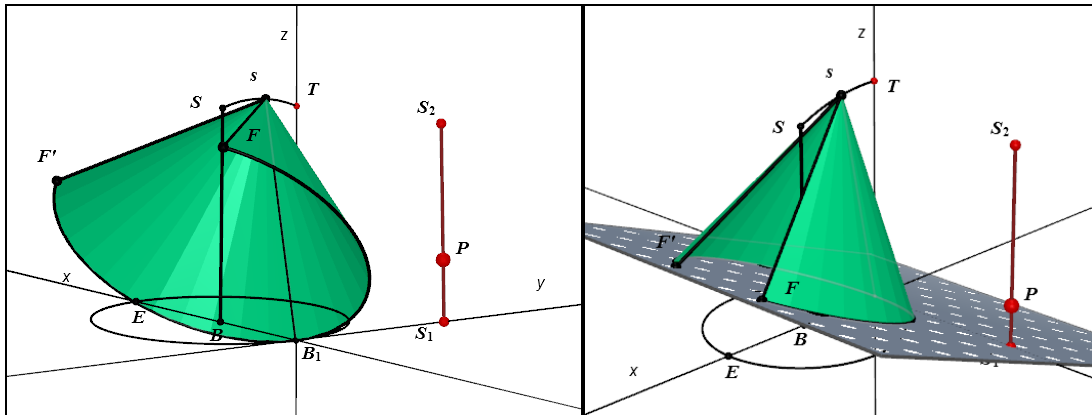


Figure 10

## 4. Some curves generated with our model

### 4.1. Trajectories of points F and F' during the unfolding process (figure 11)

We unfold the cone when  $s$  is on  $S$ ; we obtain two curves in 3D connected in E after activating the trajectory of F and F' and dragging point P along the segment  $[S_1S_2]$  (figure 11 on the left). We can check that these curves are naturally included in the sphere centred in S with SE as a radius (figure 11 on the right).



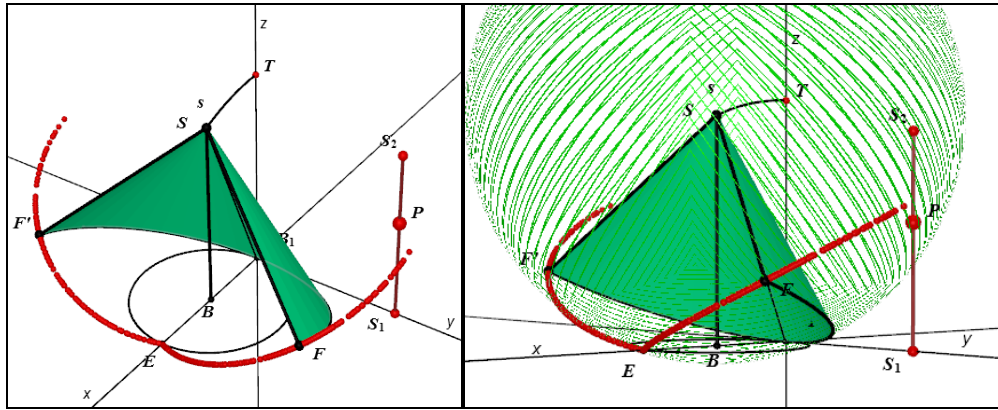


Figure 11

In figure 12 we unfold the cone when  $s$  is on  $T$ . The trajectories of  $F$  and  $F'$  are more than the previous ones rotated around the  $y$  axis (figure 12 on the left). In reality, we have explored another situation: we have redefined point  $P$  on the line  $(S_1S_2)$ , so we can continue to unfold mathematically our cone behind the  $yB_1z$  plane and the trajectories of  $F$  and  $F'$  are extended. We have displayed their projections on the  $xB_1y$  plane to obtain a curve which will be interesting to investigate (figure 12 on the right). It is another problem for another paper.

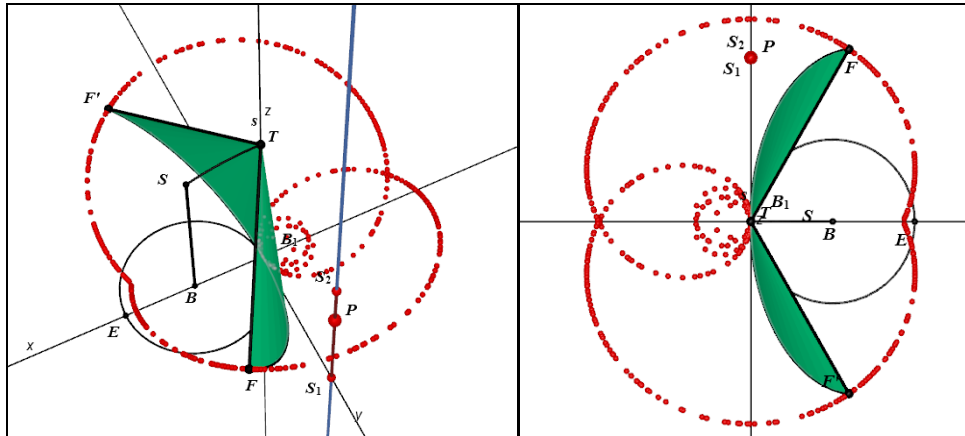


Figure 12

#### 4.2. Volume of the convex hull of the unfolded cone (figure 13)

The tool “Convex polyhedron” in Cabri 3D allows us to construct the convex hull of the unfolded pyramid (figure 13 on the left). We can now investigate the variations of the volume of this hull with respect to the variable  $S_1P/S_1S_2$ . Using the tools “measurement transfer” and “trajectory” , we display the curve of this function (figure 13 on the middle). We observe that this volume increase from the volume of the initial cone to a maximum and decrease until 0 when the unfolding process is finished. If we increase the number of digits of the displayed numbers to 4, we can investigate accurately for what value of  $S_1P/S_1S_2$  we reach this maximum (one particular case is shown in figure 13 on the right). By now, I haven’t found any interesting result even when I tried to find a relation between this maximum and the value of  $\frac{r}{h}$ . In the next paragraph, I give nevertheless the function explaining the volume of this convex hull. The important thing in this exploration is that

Cabri allows the production of data and we know that the production of data is the beginning of an experiment whose role is to lead to a conjecture. A good research can lead to a discovery but it can open a window on new problems to explore.

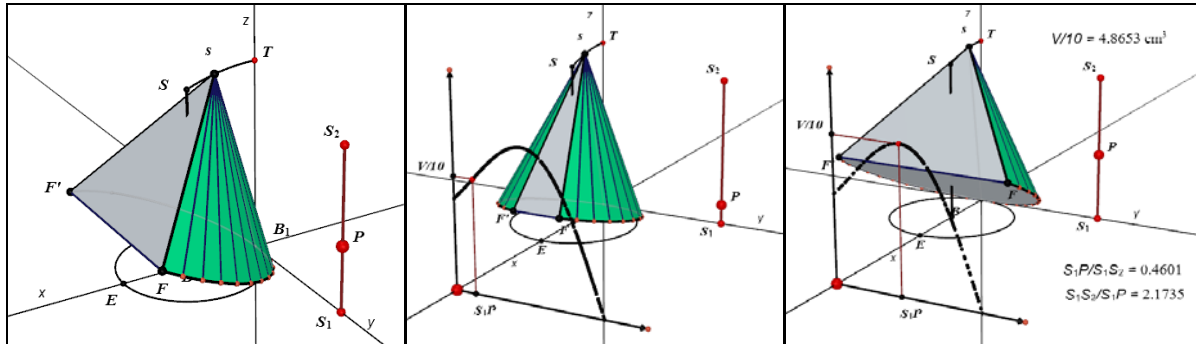


Figure 13

#### 4.3. Volume of the convex hull of the unfolded cone (figure 13)

The result obtained in 3.3. leads to another model of the (un)folding of the cone (figure 14 on the left). This new model will help us to find the function giving the volume of the previous convex hull. The new way of unfolding the cone is to move the segment  $sB_1$  around the  $y$  axis and keep the basis in contact with the  $xOy$  plane. We know that this contact is an arc included in the circle intersection between the  $xOy$  plane and the sphere centred in  $s$  and passing through  $B_1$ . The arc is defined with a measurement transfer of  $\pi \cdot r$  on this circle (from  $B_1$ ).

A pedagogical consequence: we can create an experiment with middle school students with paper and pencil. We can propose them to start with a cone created with its net in paper (posed on a table), to flatten this net like in figure 14 (on the left), to draw the curve in contact with the table and to conjecture its nature (it will be an arc included in a circle centred on the point which is the projection on the table of the point  $s$ ).

The volume of the convex hull will be given by the volume of a part of a cone, plus or minus the volume of a prism. The two possibilities are displayed in figure 14 (on the middle and on the right).

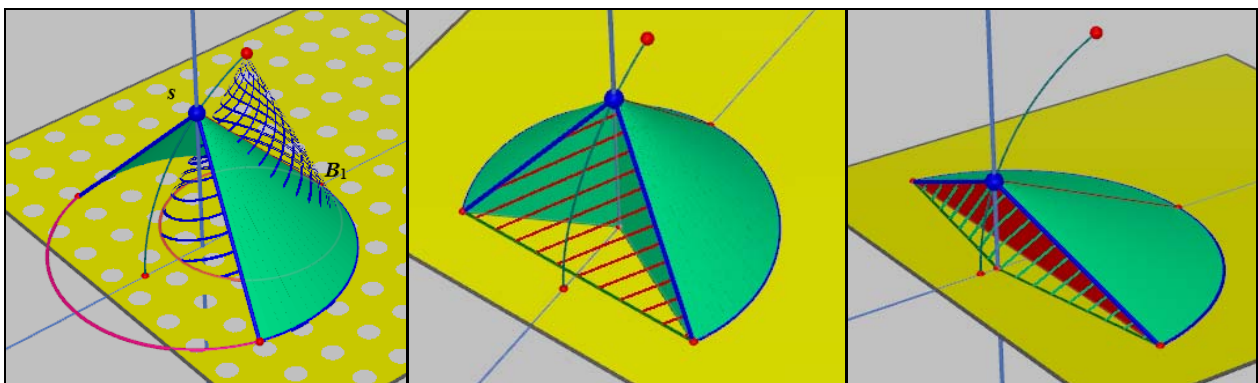


Figure 14

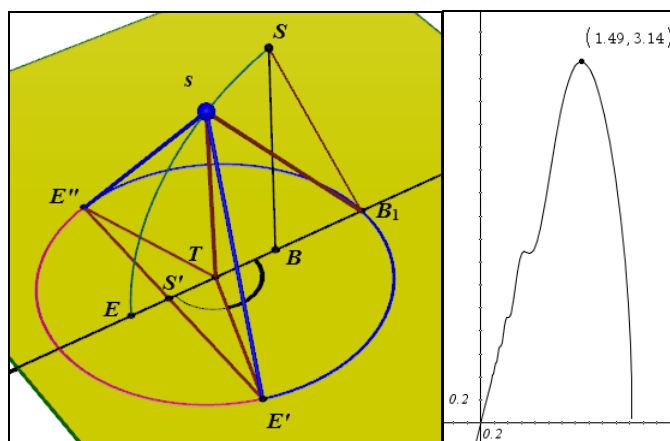
On figure 15 (on the left), we choose as a variable the distance  $B_1T = t$ ;  $t$  varies between  $r$  and  $\sqrt{r^2 + h^2}$  (which is  $SB_1$  or  $EB_1$ ). The angle  $\angle B_1TE'$  is  $\frac{r}{t}\pi$ , so the angle  $\angle S'TE'$  is  $\frac{t-r}{t}\pi$ .



$$V_1(t) = \frac{r}{t} \frac{\pi t^2 \sqrt{r^2 + h^2 - t^2}}{3} = \frac{\pi r t \sqrt{r^2 + h^2 - t^2}}{3}.$$

$V_2(t) = \frac{1}{3} \cdot \frac{t^2}{2} \sin(\frac{t-r}{t} 2\pi) \cdot \sqrt{r^2 + h^2 - t^2}$  because the area of triangle E'TE'' is  $\frac{t^2}{2} \sin(\frac{t-r}{t} 2\pi)$  and the height of the prism sT is  $\sqrt{r^2 + h^2 - t^2}$ . At last, in any case, the volume of the convex hull is given by the formula:  $V(t) = V_1(t) + V_2(t)$  where  $V_2(t)$  can be negative when  $h > r\sqrt{3}$ .

$$V(t) = \frac{1}{3} \sqrt{r^2 + h^2 - t^2} (\pi r t + \frac{t^2}{2} \sin(\frac{t-r}{t} 2\pi))$$



Now we can understand that it is difficult to find the value of  $t$  such as  $V'(t) = 0$ . The formula of  $V'(t)$  obtained with the CAS application of TI Nspire is displayed below in the figure 15.

$$\left( \frac{-\sin\left(\frac{2\cdot\pi\cdot r}{t}\right)\cdot t}{3} + \frac{\pi\cdot\cos\left(\frac{2\cdot\pi\cdot r}{t}\right)\cdot r}{3} + \frac{\pi\cdot r}{3} \right) \cdot \sqrt{-t^2+r^2+h^2} + \frac{t^2\cdot\left(\sin\left(\frac{2\cdot\pi\cdot r}{t}\right)\cdot t - 2\cdot\pi\cdot r\right)}{6\cdot\sqrt{-t^2+r^2+h^2}}$$

But the same CAS cannot solve the equation  $V'(t) = 0$ .

We have also used the Graphs & Geometry application of TI NSpire to visualize the curve of the function  $V$  (figure 15 on the right) for some particular values of  $h$  ( $h = 2$ ) and  $r$  ( $r = 1$ ). This curve validates the conjecture we did about the variations of function  $V$ . It would be interesting to continue the exploration in changing the values of  $h$  and  $r$  with a slider which is possible in this environment to try to discover some relationship between these values and the maximum of the volume of the convex hull. It could be the starting point of another research for another paper.

## 5. Conclusion

As a response to the needs of a middle school teacher, we have modelled the (un)folding of a cone with the (un)folding of a  $2n$ -regular pyramid ( $n=12$  in this paper) in Cabri 3D; we have used some mathematical tricks allowed especially by the tools “Rotation”, “Perpendicular Bisector” and “Reflection in a Plane” in Cabri 3D. The improvement of the design of the model has led to a realistic tool for our middle school teacher. We have shown how she could use it to convince her students that the net of a cone is not necessarily half a disk. We have used it to explore new problems in relation to this net and solve some of them after conjecturing the results dynamically. We have explored a problem related to the volume of the convex hull of our unfolded net: we haven’t solved this problem but the importance of this unsuccessful process is to show that a problem is a good problem not because we solve it but often when this problem opens a windows on new interesting problems. Thanks to this research we have created a second modelling of the (un)folding of a cone (in 4.3. with  $n=24$ ) and also an experimental activity for the middle school students using paper and pencil. This paper aims to show the power of experimentation in mathematics, especially when experiments are conducted with ICT, here with Cabri 3D.

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