Integrating Computers into Mathematics classes in a Unique way – Classroom Examples

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"In mathematics instruction programs, technology should be used widely and responsibly, with the goal of enriching students' learning of mathematics" (NCTM, 2000).

- ***In our efforts to integrate computers into mathematics classes and expose the students to new teaching methods, we developed two technology based courses.***
- ***These courses are taught to mathematics B.Ed and M.Ed students in a teacher training college.***

Our aim is to:

- ***Provide tools to solve mathematical problems***
- ***Adjust Mathematics Teaching and Learning to Technological Changes***
- ***Improve students' math understanding***
- ***Motivate students to learn mathematics***

Our aim is to: (continue)

- ***Raise students' mathematical curiosity as to how the computer functions***

How does the computer do it?

How does he know?

The students get acquainted with the mathematical ideas and numerical methods embedded in the computer, calculator and graphic calculator.

In other words, they learn "the story behind the key".

two major subjects of the course are:

- *Calculating the digits (to a desired accuracy) of irrational numbers ($e, \pi, \sqrt{2}$)*

  In this presentation we focus on:

  finding the square and cubic roots of a given number.

- *solving equations (one of the oldest subjects in math)*

  Two methods will be presented

We present:

- ***Heron's method***
- ***The intuitive 'trial and error' method***

  For computing the square root and the cubic root

  (do not require profound math knowledge)

- ***Bisection method***
- ***Newton Raphson Method***

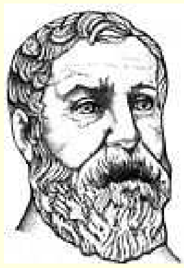
  The last 2 methods are very powerful for solving equations in general

Heron's iterative formula for computing the square root of s (a given positive number)

His method was based on:

Getting a sequence of rectangles, all with area S, so that both sides are getting closer to each other. As a limit of the sequence we get a square. The side of which is the desired square root of S.

Microsoft Excel Worksheet



Heron of Alexandria 100 a.d.

- ***At first the students use calculators and see that using Heron's method yields the desired root quite quickly.***

- ***Then they write an algorithm and translate it to a computer program using excel, realizing the "strength" of computers (generalization for every square, quick and easy way to get the answer).***

- ***They construct a permanent software that is both efficient and fully automatic.***

The students generalize-

Computing the <u>cubic root</u> of a given number

This method is based on:

Getting a sequence of parallelepipeds all with volume V and a base which is a square with sides m.

The height h is getting closer to the base side m in each iteration.

As a limit of the sequence we get a cube. The sides of which are the desired cubic roots of V.

Microsoft Excel Worksheet

The intuitive 'trial and error' method

Based on finding 2 sequences of upper and lower bounds which get closer and closer to the root, until the desired accuracy is reached.

Microsoft Excel Worksheet

Done in a similar way as making a <u>binary search.</u>

שורש שלישי של 72

שיטת הרון

a	h	v
1	72	72
24.66667	0.118335	
16.48389	0.26498	
11.07759	0.586735	
7.580636	1.252914	
5.471395	2.405118	
4.449303	3.637044	
4.17855	4.123645	
4.160248	4.160006	
4.160168	4.160168	

בדיקה

4.160168

שיטה של ניסוי וטעיה

a	b	x	x^3	v
4	5	4.5	91.125	72
4	4.5	4.25	76.76563	
4	4.25	4.125	70.18945	
4.125	4.25	4.1875	73.42847	
4.125	4.1875	4.15625	71.79678	
4.15625	4.1875	4.171875	72.60957	
4.15625	4.171875	4.164063	72.20241	
4.15625	4.164063	4.160156	71.99941	
4.160156	4.164063	4.162109	72.10086	
4.160156	4.162109	4.161133	72.05012	
4.160156	4.161133	4.160645	72.02476	
4.160156	4.160645	4.1604	72.01209	
4.160156	4.1604	4.160278	72.00575	
4.160156	4.160278	4.160217	72.00258	
4.160156	4.160217	4.160187	72.00099	
4.160156	4.160187	4.160172	72.0002	
4.160156	4.160172	4.160164	71.9998	
4.160164	4.160172	4.160168	72	
4.160164	4.160168	4.160166	71.9999	
4.160166	4.160168	4.160167	71.99995	
4.160167	4.160168	4.160167	71.99998	
4.160167	4.160168	4.160167	71.99999	
4.160167	4.160168	4.160168	72	
4.160168	4.160168	4.160168	72	

comparison

שורש שלישי של 45

שיטת הרון

a	h	v
1	45	45
.66667	0.183341	
.50556	0.407731	
139616	0.882801	
054011	1.761733	
956585	2.874561	
59591	3.480125	
557315	3.55605	
556893	3.556893	

בדיקה

3.556893

שיטה של ניסוי וטעיה

a	h	x	x^3	v
3	4	3.5	42.875	45
3.5	4	3.75	52.73438	
3.5	3.75	3.625	47.63477	
3.5	3.625	3.5625	45.21313	
3.5	3.5625	3.53125	44.03372	
3.53125	3.5625	3.546875	44.62083	
3.546875	3.5625	3.554688	44.91633	
3.554688	3.5625	3.558594	45.06457	
3.554688	3.558594	3.556641	44.99041	
3.556641	3.558594	3.557617	45.02748	
3.556641	3.557617	3.557129	45.00894	
3.556641	3.557129	3.556885	44.99968	
3.556885	3.557129	3.557007	45.00431	
3.556885	3.557007	3.556946	45.00199	
3.556885	3.556946	3.556915	45.00083	
3.556885	3.556915	3.5569	45.00026	
3.556885	3.5569	3.556892	44.99997	
3.556892	3.5569	3.556896	45.00011	
3.556892	3.556896	3.556894	45.00004	
3.556892	3.556894	3.556893	45	
3.556892	3.556893	3.556893	44.99998	
3.556893	3.556893	3.556893	44.99999	

	A	B	C	D	E	F	G	H	I	J
1										
2							שורש שלישי של 45			
3										
4	שיטת הרון						שיטה של ניסוי וטעיה			
5										
6	a	h	v			a	h	x	x^3	v
7	1	=$C$7/(A7^2)	45			3	4	=(F7+G7)/2	=H7^3	45
8	=(2*A7+B7)/3	=$C$7/(A8^2)				=IF(I7<$J$7,H7,F7)	=IF(I7>$J$7,H7,G7)	=(F8+G8)/2	=H8^3	
9	=(2*A8+B8)/3	=$C$7/(A9^2)				=IF(I8<$J$7,H8,F8)	=IF(I8>$J$7,H8,G8)	=(F9+G9)/2	=H9^3	
10	=(2*A9+B9)/3	=$C$7/(A10^2)				=IF(I9<$J$7,H9,F9)	=IF(I9>$J$7,H9,G9)	=(F10+G10)/2	=H10^3	
11	=(2*A10+B10)/3	=$C$7/(A11^2)				=IF(I10<$J$7,H10,F10)	=IF(I10>$J$7,H10,G10)	=(F11+G11)/2	=H11^3	
12										
13				בדיקה						
14				=C1^(1/3)						

אלגוריתם לפי "ניסוי וטעיה"

1. התחל.
2. קלוט $v>0$ ($v=$ המספר שאת השורש שלו רוצים לחשב)
3. קבע a, h כך ש $a < \sqrt[3]{v} < h$.
4. כל עוד $a \neq h$ (בדיוק הרצוי) בצע:
   - 4.1 $x \leftarrow \frac{a+h}{2}$.
   - 4.2 חשב x^3.
   - 4.3 אם $x^3 < v$ אז $a \leftarrow x$
     אחרת $h \leftarrow x$.
   - 4.4 בכל שלב הדפס את a, h, x, x^3.
5. אחרת הדפס a, h, x, x^3.
6. סיים.

אלגוריתם לפי שיטת הרון

1. התחל
2. קלוט $v>0$ (v המספר שאת השורש השלישי שלו רוצים לחשב)
3. $a \leftarrow 1$
4. $h \leftarrow \frac{v}{a^2}$
5. כל עוד $a \neq h$ (בדיוק הרצוי) בצע:
   - 5.1 $a \leftarrow \frac{2a+h}{3}$
   - 5.2 $h \leftarrow \frac{v}{a^2}$
   - 5.3 בכל שלב הדפס a, h
6. אחרת הדפס a, h
7. סיים.

Solving equations

How can we obtain solutions for any desired accuracy.

Students are not aware that:

There does not exist (and will never be found) a closed formula for solving polynomial equations of an order greater than 4 (Abel, Galois, Lie), and for other non algebraic equations.

How do we solve?

How do graphic calculators, and computer software ...know?

solving equations

To solve $f(x)=0$ (to find the real roots of the equation)

we look for the zeros of the (continuous) function $y=f(x)$

We focus first on the equation $x^2-s=0$

$$x^2-s=0 \iff \begin{cases} f(x)=x^2-s \\ f(x)=0 \end{cases}$$

Bisection method

- Choose a relevant interval [a,b] where $f(a)=a^2-S<0$ and $f(b)=b^2-S>0$

  The required value of the positive square root of S lies between a and b (Cauchy's mean value theorem),

  precisely where the graph of the parabola intersects the x axis.

Finding one positive root for an increasing function

Choose a relevant interval [a,b] where

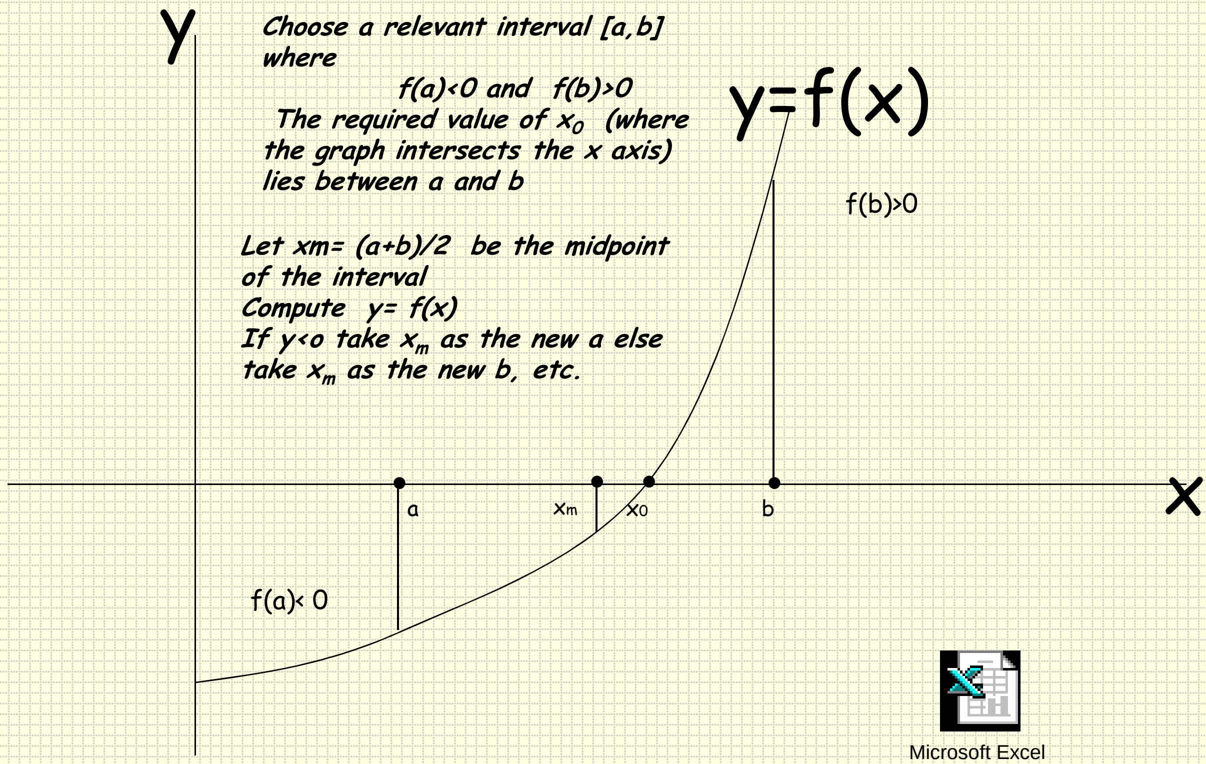
f(a)<0 and f(b)>0

The required value of x_0 (where the graph intersects the x axis) lies between a and b

Let xm= (a+b)/2 be the midpoint of the interval

Compute y= f(x)

If y<o take x_m as the new a else take x_m as the new b, etc.



Microsoft Excel Worksheet

- ***The equations with which we deal have no simple closed formula for their roots, as the quadratic equation has.***
- ***We turn to methods of approximating the real roots to some prescribed degree of accuracy.***

Examples for ***solving equations*** *using* ***Bisection*** *method*

In order to solve each of the following equations, Investigate the appropriate function, decide the number of zeros and plot. check with software:

- $x^3+2x^2+10x-20=0$ **(Fibbonacci, 1225, x=1.36880810)**
- $xe^{-x}-0.25=0$
- $2^x+x-2=0$
- $Sinx-x/2=0$
- $X^7+2x-200=0$

Microsoft Excel Worksheet

Newton Raphson Method

(1690)

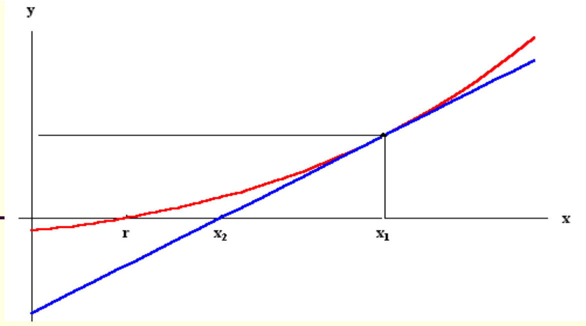
- ***finding the roots of*** $f(x)=0$

  (a differentiable function)

- ***or finding the zeros of*** $y=f(x)$

Newton:1643-1727

Using the tangent line



- ***Find the number of zeros***
  (Using calculus and/or software)
- ***For each of the zeros find a first approximation x_1 and the point A***
- ***Find a=f'(x_1)***
- ***For each iteration compute***

$$\boxed{x_{n+1}=x_n-f(x_n)/f'(x_n)}$$

$f'(x_n)\neq 0$

Solving the equation $x^2-s=0$

using Newton Raphson's method

yields the same formula (and result)

as

Heron got without using calculus.

Examples for solving equations using Newton-Raphson's method

Solving the same equations, yields a much quicker solution (second order method)

- $x^3+2x^2+10x-20=0$ (Fibbonacci, 1225, x=1.36880810)
- $xe^{-x}-0.25=0$
- $2^x+x-2=0$
- $Sinx-x/2=0$
- $X^7+2x-200=0$

n-r.xls

During the course

- *The students have learned many and varied numerical methods taken from different branches of mathematics.*
- *Emphasis is given to the mathematical knowledge and to accompanying justifications.*

- ***The students deal with new and vital subjects (taken from Discrete algorithmic Mathematics and Numerical Analysis),***
- ***These topics are ordinarily learned in advanced undergraduate mathematics courses or in Computer Science studies***
- ***They are absent from the regular curriculum in schools in Israel.***

Technological developments make it possible to incorporate selected chapters of these two courses earlier, in high school or even in the upper grades of elementary school curriculum, by adapting the topics to students' knowledge.

It should be pointed out that

In this presentation we showed only a glimpse of what we teach in the courses and how we integrate computers into mathematics classes.

We hope that

- ***These topics will be integrated into the curriculum***
- ***Our students will be the agents who incorporate it into schools.***
- ***This way of teaching will contribute to raise the next hi- tech generation.***

THANK YOU