# Analogy and dynamic geometry software together in approaching 3d geometry<sup>1</sup>

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#### ABSTRACT

We present a didactic proposal on Euclidean geometry, both plane and space geometry, finalized to make three dimensional geometry more catchy and simple. The proposal consists of a guided research activity that leads the students to discover unexpected properties of two apparently distant geometrical entities, that is, quadrilaterals and tetrahedra. The activity has been realized by means of an efficacious conceptual tool, the *analogy*, and an operative one, a *dynamic geometry software*.

#### Keywords

Quadrilaterals, tetrahedra, analogy, dynamic geometric software, mathematical laboratory.

#### Introduction

Geometry is an irreplaceable conceptual tool that describes the world we live in and turns in a privileged environment where to learn how to express and argue your own thoughts.

In high school programming, geometry, and in particular space geometry, is always more often unfairly neglected and consequently students are deprived not only of fundamental geometry knowledge but also of important geometric cognitive stimulus that are important in the formative process. We do not have to forget the fundamental function that geometry has in developing competencies to relate language, thought and ability in building up rigorous reasoning.

Three dimensional geometry is, for sure, more complex than two the dimensional one. In fact, it presents difficulties of conceptual type as well as difficulties in realizing and interpreting the drawing in two dimensions of three dimensional figures: a draw in two dimensions of a three dimensional figure can not be faithful because it is not possible to save all the lengths of segments

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and the width of angles.

For these difficulties teachers resize or even delete three dimensional geometry in high school teaching. Students are then deprived of the possibility of developing their ability of spatial visualization.

Firmly convinced that "a study of plane geometry without flowing into space one is like getting ready to playing a match knowing not to pass it" [24], we deem that it is necessary to find appropriate didactic strategies finalized to encourage students not to give up at first difficulties and to help them to overcame obstacles.

For this purpose it could be appropriate dealing with plane geometry and space geometry at the same time. Villani suggests to refer to "an attenuated form of what called in the late Eighteen century and the beginning of Nineteen one "fusionism" between plane and space geometry consisting in teaching in parallel bi- and tri-dimensional one. For example, mutual positions among lines in the plane and planes in the space, plane angles, dihedral angles and polyhedral angles, parallelograms and parallelepipeds and, more in general, polygons and polyhedra, circumferences and spheres, areas and volumes, Pythagoras's and Talete's theorems and their possible extensions to the three dimensional case" [24]. Besides, even without changing to the traditional teaching, that is, starting from plane geometry- to allude to the analogies and differences with space geometry and, successively, while following the program of the second one –space geometry - to connect the three dimensional notions with the correspondent two dimensional ones, that have already been studied, so to mark analogies and differences.

#### Our approach: analogy and software in a mathematical teaching laboratory

Within the theoretical frame we just talked about, in order to simplify the approach to three dimensional geometry, we believe that is opportune make students closer to three dimensional geometry by presenting them subjects with strong analogies with plane subjects. In the activity is good to mark how often, working in the space, you end-up in working in the plane.

A team of university professors and high school teachers has developed a laboratory activity that aims to connect the plane and the space [9]. The chosen topic is the one of quadrilaterals and tetrahedra: we start with quadrilaterals, already familiar to the students, and study tetrahedra utilizing the numerous analogies with quadrilaterals. The proposal is based on the paper [12].

We want to help students in using concepts previously studied in order to discover new properties and to compare plane and space figures for searching analogies between situations apparently different. A new and catchy approach to three dimensional geometry has been realized by means of an efficacious conceptual tool, the *analogy*, and an operative one, a *dynamic geometry software*:

• The *analogy* is "a sort of similarity among distinct objects. Similar objects agree with each other in some aspects, analogous objects agree in clearly definable relations of their respective parts" [19]. To solve a problem you can use the solution of an analogous simpler problem end, of this last problem, use its method, its result or both method and result. In our case, we underline a strong analogy between quadrilaterals and tetrahedra and we use both, results and method used to study some properties of quadrilaterals, whose resolution has then been traced step by step in facing analogous properties of tetrahedra. The use of the analogy turns out to be precious because it represents a bridge that creates a significant link between two and three dimensions. Analogy, in fact, not only makes the understanding of properties of tetrahedra easier but also has an important rule in overcoming difficulties in visualizing solids.

• The dynamic geometry software, Cabri Géomètre in our case, helps us not only to easily make plane and space geometric figures, but also to dynamically change them without modifying the properties used for building them. With the use of Cabri Géomètre we retrieve the manipulative aspect, and the intuitive one, that is of the previous learning processes [23], that turn out to be essential to get to a level of rational thought. In particular, Cabri 3D, the three dimensional version of Cabri Géomètre, can be a really useful instrument to overcome problems inherent the visualization of three dimensional figures. In fact, this software allows us to examine the objects from different perspectives by changing the point of view (see for example [22]).

We want to realize our proposal in a "mathematical teaching laboratory", intended as "a phenomenological space to teach and learn mathematics developed by means of specific technological tools and structured negotiation processes in which maths knowledge is subjected to a new representative, operative and social order to become object of investigation again and be efficaciously taught and learnt" [5].

The laboratory as a mathematics teaching and learning environment is today often used [1, 7, 13, 14, 16, 20, 21] and also the Italian Mathematics Union, in writing the new curricula, suggests [18]: "We can imagine the laboratory environment as a Renaissance workshop, in which the apprentices learned by doing, seeing, imitating, communicating with each other, in a word: practising. In the laboratory activities, the construction of meanings is strictly bound, on one hand, to the use of tools, and on the other, to the interactions between people working together [...] to the

communication and sharing of knowledge in the classroom, either working in small groups in a collaborative and cooperative way, or by using the methodological instrument of the mathematic discussion, conveniently lead by the teacher".

In the laboratory students explore a problem that they can solve: it has to be not too hard neither too easy, not boring but challenging. They have to have all the knowledge they need and their work has to be guided from the teacher. The teacher has a crucial role. He has to guide the pupils to attain various results by way of trial and error, to direct the students with appropriate suggestions on the path to follow, to question the proposals that still need to be perfected using counter examples, to encourage them to continue, to praise them for every significant result. Moreover, he beats time and create the right atmosphere.

We believe that such activities are an involving way of doing geometry, finalized to reinforce representing/drawing skills and abilities in exploring geometry situations, to promote autonomous production of conjectures and to stimulate the need of elaborating a rational argumentation that turns into the proof of what was just discovered.

#### **Our proposal**

Our proposal enters the educational route of space geometry in high school.

In literature traingles and tetrahedra are often assimilated because they are poligons of the plane and polyhedra of the space with the least number of vertices. In our proposal tetrahedra are considered in analogy with quadrilaterals [12]. The analogy comes from the fact that quadrilaterals are defined as figures of the plane determined by four vertices, such that any three of them are non-collinear, and with six edges (the four sides and the two diagonals) and four faces (the triangles determined by three vertices of the quadrilateral), and tetrahedra are defined as figures of the space determined by four non coplanar, and with six edges and four faces.



In both figures we introduce analogous definitions (bimedian, centroid, median, axis/axial plane, circumcentre, maltitude/Monge plane, anticentre/Monge point) and prove analogous properties. For quadrilaterals refer to [6, 8, 10, 11, 12, 17] and for tetrahedra to [2, 4, 12].

In the activity, by using Cabri 3D, students pass from quadrilaterals to tetrahedra by "extracting" a vertex of the quadrilateral in the space. This "operation" is often repeated in the proposal when looking for analogies between the two objects.

A final table ricapitulates all the results that have been obtained and marks all the analogies.

QUADRILATERALS	TETRAHEDRA
<b>Q</b> is a <b>convex quadrilateral</b> with <b>vertices</b> A, B, C, D.	<b>T</b> is a <i>tetrahedron</i> with <i>vertices A, B, C, D</i> .
The points A, B, C, D are such that any three of them are non-collinear.	The points A, B, C, D are non coplanar.
The vertices detect six segments AB, BC, CD, DA, AC, BD, that are called <i>edges</i> . The edges of $Q$ are the four sides and the two diagonals.	The vertices detect six segments AB, BC, CD, DA, AC, BD, that are called <i>edges</i> .
Two <i>edges</i> are said to be <i>opposite</i> if they do not have common vertices. They are opposite edges: AB and CD, BC and DA, AC and BD, that is, either two opposite sides or the two diagonals.	Two <i>edges</i> are said to be <i>opposite</i> if they do not have common vertices. They are opposite edges: AB and CD, BC and DA, AC and BD.
We call <i>faces</i> of $Q$ the triangles determined by three vertices of $Q$ . There are four faces: ABC, BCD, CDA, DAB.	We call <i>faces</i> of $T$ the triangles determined by three vertices of $T$ . There are four faces: ABC, BCD, CDA, DAB.
A <i>vertex</i> and a <i>face</i> are said to be <i>opposite</i> if the vertex does not belong to the face. For each vertex there is one and only one opposite face.	A <i>vertex</i> and a <i>face</i> are said to be <i>opposite</i> if the vertex does not belong to the face. For each vertex there is one and only one opposite face.
The segment joining the midpoints of two opposite edges of $Q$ is called <i>bimedian</i> of $Q$ .	The segment joining the midpoints of two opposite edges of $T$ is called <i>bimedian</i> of $T$ .
Q has three bimedians, two relative to a pair of opposite sides and one relative to the	<i>T</i> has three bimedians.

#### TABLE OF THE ANALOGIES

diagonals.	
Theorem 1. The three bimedians of a quadrilateral all pass through one point.	Theorem 1. The three bimedians of a tetrahedron all pass through one point.
The point G common to the three bimedians of $Q$ is called the <i>centroid</i> of $Q$ .	The point G common to the three bimedians of $T$ is called the <i>centroid</i> of $T$ .
Theorem 2. The centroid bisects each bimedian.	Theorem 2. The centroid bisects each bimedian.
The segment joining a vertex of $Q$ with the centroid of the opposite face is called <i>median</i> of $Q$ . $Q$ has four medians.	The segment joining a vertex of $T$ with the centroid of the opposite face is called <i>median</i> of $T$ . $T$ has four medians.
Theorem 3. The four medians of a quadrilateral meet in its centroid.	Theorem 3. The four medians of a tetrahedron meet in its centroid (Commandino's Theorem).
Theorem 4. The centroid of a quadrilateral divides each median in the ratio 1:3, the longer segment being on the side of the vertex of Q.	Theorem 4. The centroid of a tetrahedron divides each median in the ratio 1:3, the longer segment being on the side of the vertex of T.
Theorem 5. The quadrilateral of the centroids of the faces of a quadrilateral $Q$ is the image of $Q$ with dilatation ratio $-1/3$ and center the centroid of $Q$ .	Theorem 5. The tetrahedron of the centroids of the faces of a tetrahedron T is the image of T with dilatation ratio $-1/3$ and center the centroid of T.
The line that is perpendicular to an edge of $Q$ in its midpoint is called <i>axis</i> of the edge. $Q$ has six axes.	The perpendicular plane to an edge of <i>T</i> in its midpoint is called <i>axial plane</i> of the edge. <i>T</i> has six axial planes.
Theorem 6. The axes of the edges of a cyclic quadrilateral meet in a point.	Theorem 6. The axial planes of the edges of a tetrahedron meet in a point.
The common point to the axes of a cyclic quadrilateral $Q$ , i.e. the centre of the circle circumscribed to $Q$ , is called <i>circumcenter</i> of $Q$ .	The common point to the axial planes of a tetrahedron $T$ , i.e. the centre of the sphere circumscribed to $T$ , is called <i>circumcenter</i> of $T$ .
The line that is perpendicular to an edge of a quadrilateral $Q$ and passes through the midpoint of the opposite edge is called <i>maltitude</i> of $Q$ . $Q$ has six maltitudes.	The plane that is perpendicular to an edge of a tetrahedron $T$ and passes through the midpoint of the opposite edge is called <b>Monge plane</b> of $T$ . $T$ has six Monge planes.
Theorem 6. The maltitudes of a cyclic quadrilateral are concurrent.	Theorem 6. The Monge planes of a tetrahedron are concurrent. (Monge Theorem).
The common point to the six maltitudes of a	The common point to the six Monge planes

cyclic quadrilateral $Q$ is called <i>anticenter</i> of $Q$ .	of a tetrahedron <b>T</b> is called <b>Monge point</b> of <b>T</b> .
Theorem 7. In a cyclic quadrilateral the anticenter is symmetric to the circumcenter with respect to the centroid.	Theorem 7. In a tetrahedron the Monge point is symmetric to the circumcenter with respect to the centroid.
Theorem 8. In a cyclic quadrilateral the anticenter, the circumcenter and the centroid are collinear.	Theorem 8. In a tetrahedron the Monge point, the circumcenter and the centroid are collinear.
The line containing the anticenter, the circumcenter and the centroid of a cyclic quadrilateral $Q$ is called <i>Euler line</i> of $Q$ .	The line containing the Monge point, the circumcenter and the centroid of a tetrahedron $T$ is called <i>Euler line</i> of $T$ .

The proposal is organized in forms, five on two dimensional geometry and five on three dimensional geometry.

We used the same methodology we used in other projects already realized and tested [1, 3, 15]: we decided to offer students a path presented through forms that have been written on purpose so to make easier the learning material and bring the student, step by step, through the different phases of the work.

The teaching/learning strategy that we used in the forms follows the scheme:

Explore and verify with Cabri - Conjecture - Prove

i.e., by observing and exploring the figure perceive the relations between objects, then by dragging operations experimentally verify the hypothesis and, once they are confirmed, formulate a conjecture and prove it. This trail is proposed in the plane first and then, with the use of the existing analogy, in the space.

The forms are organized so to offer an immediate correlation between quadrilaterals and tetrahedra and therefore for any Form Q, relative to a property of quadrilaterals, there is a Form T, relative to the corresponding property of tetrahedra.

The first form is a bit different from the others: it introduces the objects and the procedures that will be used next. The other forms consist of two parts each: the first part contains hints that guide the student to discover and conjecture a property, the second part contains the statement of the theorem and a guided route to prove it.

It follows, for example, Form 2: Form 2Q – Part I, Form 2Q – Part II, form 2T – Part I, Form 2T – Part  $II^2$ .

 $<sup>^{2}</sup>$  In "Form 1Q" students, by using Cabri II, draw a quadrilateral ABCD with its six edges, four faces and save the file with the name "*Quadrilateral*". In "Form 1T" students, by using Cabri 3D, in the base plane, draw a quadrilateral ABCD with its six edges, four faces and save the file with the name "*Quadrilateral in the space*".

# Form 2 Q – Part I

#### The bimedians of a quadrilateral

**Definition**. We call **bimedian** of a quadrilateral the segment joining the midpoints of two opposite edges.

1. Open, with *Cabri II*, the file saved with the name *Quadrilateral*.

2. With instrument *Label* call  $M_1$ ,  $M_2$ ,  $M_3$ ,  $M_4$ ,  $M_5$ ,  $M_6$  the midpoints of the edges AB, BC, CD, DA, AC, BD, respectively.

There are three bimedians, two relative to pairs of opposite sides and one relative to the diagonals, i.e.:



**Observation.** The points  $M_1$  and  $M_3$  are distinct because the belong to opposite edges of the quadrilateral. For the same reason  $M_2$  and  $M_4$  are distinct. Instead, the points  $M_5$  and  $M_6$  are not always distinct. In fact, since  $M_5$  and  $M_6$ are midpoints of the diagonals of the quadrilateral, they coincide if and only if the diagonals bisect each other, i.e. if and only if the quadrilateral is a .....

3. With the instrument *Segment* draw the bimedians  $M_1M_3$  and  $M_2M_4$ . With the instrument *Intersection Point(s)* draw their meeting point and call it G.

4. Draw the bimedian  $M_5M_6$  and with the instrument *Member*? verify if G belongs to it. With the mouse drag some of the vertices of the quadrilateral. The property still holds?

YES NO

5. Save this figure in a file and call it *Bimedians-Q*.

6. Draw the two segments in which any of the three bimedians is divided by the point G and find their measures with the instrument *Distance or Length*. What do you

observe? With the mouse drag some of the vertices of the quadrilateral. The property still holds?

YES

NO

Considering what you have discovered about the three bimedians and the point G, you can state that:

#### Conjecture

.....

# Form 2Q – Part II

## The bimedians of a quadrilateral

In the previous form you have discovered the following property that we will now prove:

#### Theorem 1Q

The three bimedians of a quadrilateral all pass through one point that bisects each bimedian.

Proof.

Consider a convex quadrilateral Q with vertices A, B, C, D. Let M<sub>1</sub>, M<sub>2</sub>, M<sub>3</sub>, M<sub>4</sub>, M<sub>5</sub>, M<sub>6</sub> be the midpoints of the edges AB, BC, CD, DA, AC, BD, respectively.

The bimedians  $M_1M_3$  and  $M_2M_4$  of Q meet in a point G. Consider the quadrilateral  $M_1M_2M_3M_4$ .



In the triangle ABC the segment  $M_1M_2$  joins the midpoints of the edges ..... and

....., then it is parallel to ...... . For the same reason also the segment  $M_3M_4$  is parallel to AC, because in the triangle ...... it joins the midpoints of the edges ..... and ..... . Then, for the ...... property of parallelism,  $M_1M_2$  and  $M_3M_4$  are parallel to each other. Analogously you can prove that the segments  $M_2M_3$  and  $M_1M_4$  are both parallel to BD and therefore parallel to each other.

Then the quadrilateral  $M_1M_2M_3M_4$  is a ....., and the point G, being the common point of its diagonals, divides them in ..... parts.

Now consider the third bimedian  $M_5M_6$ . Choose another bimedian so that the endpoints of the two bimedians detect a quadrilateral. You could prove, with a similar reasoning to the previous one, that they meet in their midpoint. But the bimedian that you have chosen has G as midpoint, then G is the midpoint of  $M_5M_6$  as well.

Then we can conclude that the three bimedians of Q all pass through one point that bisects them.

**Definition**. The common point G of the three bimedians of a quadrilateral is called **centroid** of the quadrilateral.

## Form 2T – Part I

## The bimedians of a tetrahedron

**Definition**. We call **bimedian** of a tetrahedron the segment joining the midpoints of two opposite edges of the tetrahedron.

1. Open, with *Cabri 3D*, the file saved with the name *Quadrilateral in the space*. In analogy with what you have done in Form 2Q (Part I), draw the midpoints  $M_1$ ,  $M_2$ ,  $M_3$ ,  $M_4$ ,  $M_5$ ,  $M_6$  of the edges AB, BC, CD, DA, AC, BD, respectively. Draw the three bimedians of the quadrilateral by choosing, with the right click of the mouse, *curve style/Dash-line style*. With the instrument *Intersection Point(s)* draw their meeting point and call it G.

2. Extract the vertex D from the plane, by using the instrument **Redefinition** (click on D and release, keep on pressing  $\hat{U}$  (the capital letter key) and move the mouse up without clicking). Remember that you can move the vertices if the tetrahedron appear "way too squeezed on the plane". In order to have a better view of the figure you can make transparent the faces of the tetrahedron (*Manipulation*, select the face, right click, *Surface Style/Empty*).



In the previous form you have discovered the following property that we will now prove:

#### **Theorem 1T**

The three bimedians of a tetrahedron all pass through one point that bisects each bimedian.

Proof.

Consider a tetrahedron with vertices A, B, C, D. Let  $M_1$ ,  $M_2$ ,  $M_3$ ,  $M_4$ ,  $M_5$ ,  $M_6$  the midpoints of the edges AB, BC, CD, DA, AC, BD, respectively. Consider the bimedians  $M_1M_3$  and  $M_2M_4$ .



In the triangle ABC the segment  $M_1M_2$  joins the midpoint of the edges ...... and ....., then it is parallel to .....; analogously, also the segment  $M_3M_4$  is parallel to AC, because in the triangle ...... it joins the midpoint of the edges ...... and ....... It follows that  $M_1M_2$  and  $M_3M_4$  are parallel and the four points  $M_1$ ,  $M_2$ ,  $M_3$ ,  $M_4$  are coplanar. For the same reason, the segments  $M_2M_3$  and  $M_1M_4$  are both parallel to BD. Then quadrilateral  $M_1M_2M_3M_4$ parallelogram, the is a because ....., and the point G, common point of the bimedian  $M_1M_3$  and  $M_2M_4$ , is their midpoint, because it is the meeting point of the two ..... You can prove, with a similar reasoning, that also the bimedians  $M_1M_3$  and  $M_5M_6$  meet in their midpoint. Since the midpoint of  $M_1M_3$  is G, G is also the midpoint of  $M_5M_6$ .

Therefore we can conclude that the three bimedians all pass through G that bisect them.

**Definition**. The common point G of the three bimedians of a tetrahedron is called **centroid** of the tetrahedron.

A final form, which the students fill in at the end of each pair of forms, aims to build a overview of the whole activity: the students will obtain the "table of the analogies".

We have already started experimenting our proposal with encouraging results. Our purpose is to continue the experimentation in other classes next school year.

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