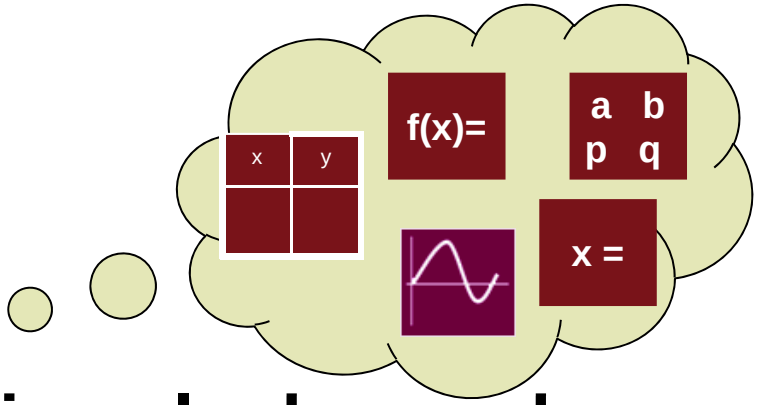
The Role of CAS when learning algebra and developing functional thinking

Bärbel Barzel,
University of Education, Freiburg



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CAS is too powerful to use it for lower secondary level - reduce its integration on upper secondary level

When using CAS pupils loose skills by hand

CAS may only be used following the white box black box principle

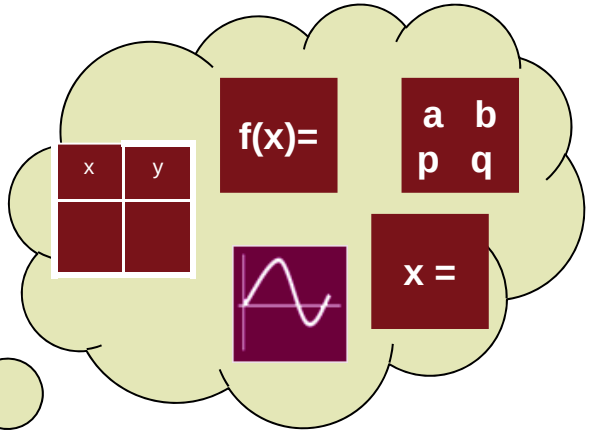
The Role of CAS when learning algebra and developing functional thinking

....

Pupils learn easier coming along with variables when CAS is integrated into the learning process

Pupils gain more skills when using CAS

CAS helps to investigate a new topic by using the black box white box principle



Focusing

1. … the topic
2. … the technology
3. … the tasks & teaching
4. … the teacher training

Algebra and functional thinking: …. problems

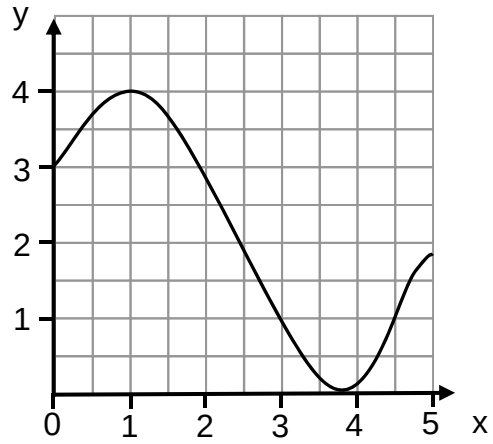
... measuring basic algebraic skills in PISA:

In which ways can „the half of the number a“ be written?
Mark with a cross if the answer is right.

$\frac{a}{2}$	Ja	☐	Nein	☐
$a - \frac{1}{2}$	Ja	☐	Nein	☐
$\frac{1}{2} \cdot a$	Ja	☐	Nein	☐
$a - \frac{a}{2}$	Ja	☐	Nein	☐
$0,5 \cdot a$	Ja	☐	Nein	☐
$a : \frac{1}{2}$	Ja	☐	Nein	☐
$\frac{1}{2} a$	Ja	☐	Nein	☐

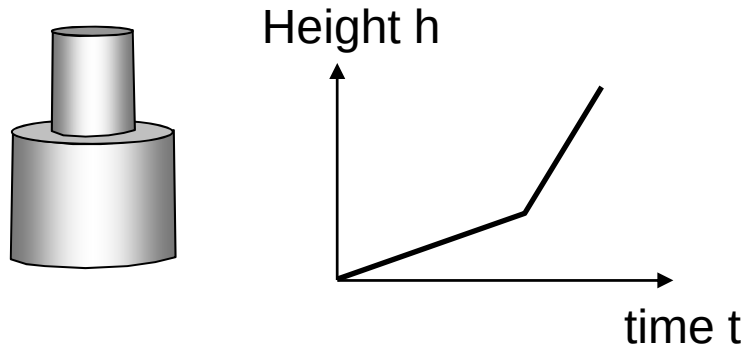
Frequency of solution in PISA: **13 %**

Algebra and functional thinking: …. problems



For which value of x is the according y-value 1?

29% wrong answer



42% sketch a wrong graph

Test-Results
1. Semester
WS 2008/09

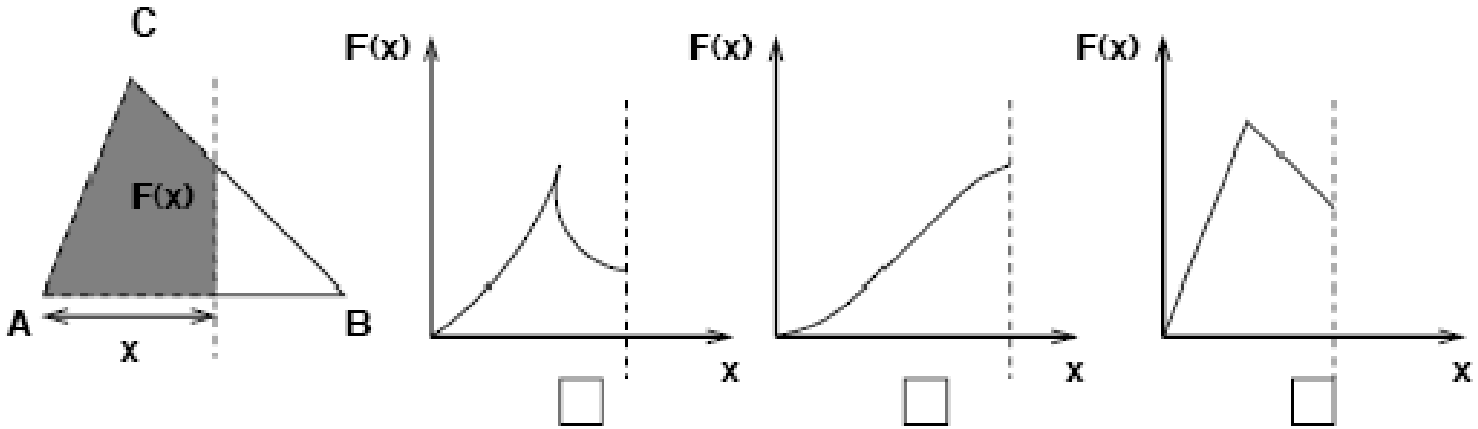
A taxi driver takes 5€ as fix costs and 3€ for every km. Sketch a graph, …

48% sketch a wrong graph

See as well: Janvier 1983, PISA 2003, PALMA 2006, Kaput 1994

Algebra and functional thinking: …. problems

Test with about 100 Math-students, 1.Semester



The dot-line is drawn from A to B.

The value *F*(*x*) gives the soze of the grey area.

a. Which graph belongs to F(x)? Make a cross.

b. Give reasons to your choice.

Results:

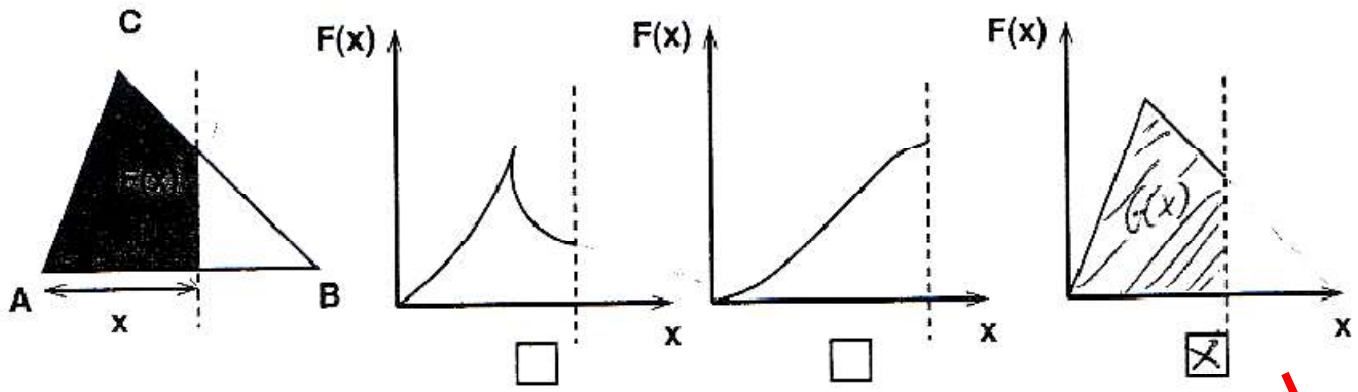
a. 66% correct answer

b. 57% correct answer

Hoffmann 2007

Algebra and functional thinking: …. problems

Test with about 100 Math-students, 1.Semester



Begründen Sie Ihre Wahl:

Die Fläche G(x) ist so, wie die Fläche F(x)

„Graph-as-Picture-irritation“

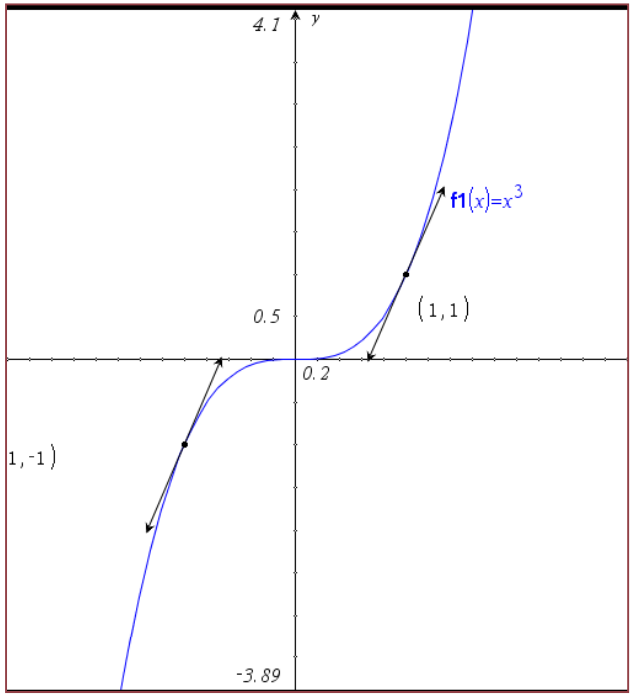
Hoffmann 2007

Algebra and functional thinking: …. problems

f is a differentiable function with *f* (-*x*) = - *f* (*x*).

What is right?

A) *f* '(-*a*) = - *f* '(-*a*)
B) *f* '(-*a*) = *f* '(*a*)
C) *f* '(-*a*) = - *f* '(*a*)
D) Keine von A-C



A typical answer was: *f* '(-*a*) = (*f* (-*a*))'= (- *f* (*a*))'= - *f* '(*a*).

Eisenberg & Dreyfus (1990)

Algebra and functional thinking: …. What is it about?

Learning basic syntactical skills	Getting a sense for symbols and structures	Dealing flexible with functions and their representations

$5\,(\,2x + 1\,) = 10$

$10x + 5 = 10$

…

$5\,(\,2x + 1\,) = 10$

$5\,(\,2x + 1\,) = 2 \bullet 5$

…

$f(u) = a \bullet u + a^2$

Jahnke 1999; Krüger 2000; Schubring 2007 ; Sierpinska 1992; Arcavi 1992

Algebra and functional thinking: …. What is it about?

Learning basic syntactical skills	Getting a sense for symbols and structures	Dealing flexible with functions and their representations

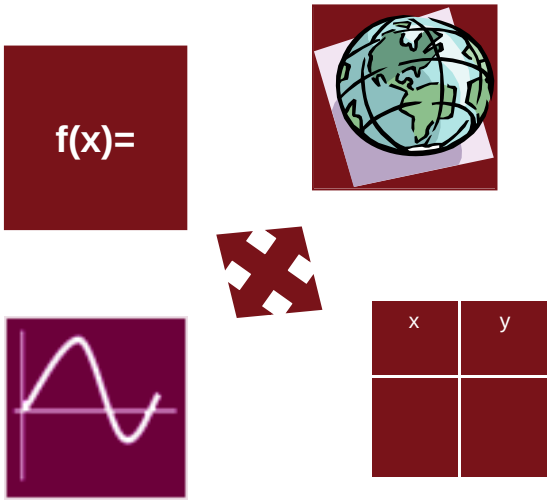
- Connecting mathematical representations (verbal, graphic and numeric representations)
- Developing and connecting different mental images

Presmeg 1997; 2006; Duval, 2002; Tall 1997; Fischer 1987; Malle 2000; vom Hofe 1995

Algebra and functional thinking: …. What is it about?

Learning basic syntactical skills	Getting a sense for symbols and structures	Dealing flexible with functions and their representations

- Connecting mathematical representations (verbal, graphic and numeric representations)
- Developing and connecting different aspects/ images of the concept



Algebra and functional thinking: …. What is it about?

Learning basic syntactical skills	Getting a sense for symbols and structures	Dealing flexible with functions and their representations

- Connecting mathematical representations (verbal, graphic and numeric representations)
- Developing and connecting different aspects/ images of the concept

´**Variable**´ can be seen

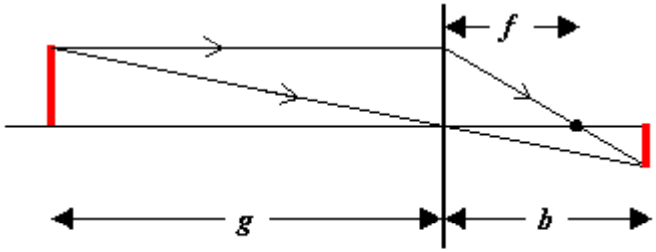
- as a placeholder
- as a generalizer
- as an unknown
- as a changing quantity

´**Function**´ can be seen:

- under the aspect of mapping
- under the aspect of covariation
- as an entity

Malle 1993 & 2000; Fujii 2003

Algebra and functional thinking: …. What is it about?



Lens equation

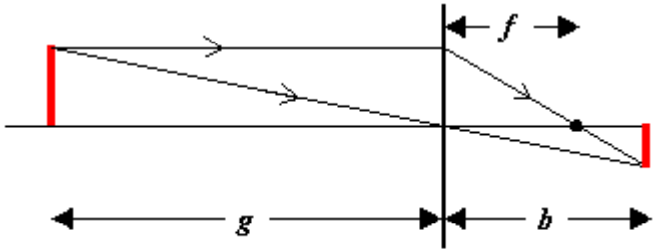
$$\frac{1}{g} = \frac{1}{f} - \frac{1}{b}$$

Drijvers 2005

Typing in correctly, otherwise:

$$\frac{1}{g} = \frac{1}{f - \frac{1}{b}}$$

Algebra and functional thinking: …. What is it about?



Lens equation

$$\frac{1}{g} = \frac{1}{f} - \frac{1}{b}$$

as a placeholder:

$$\frac{1}{g} = \frac{1}{f} - \frac{1}{b} \;\Big|\; f = 10 \qquad \frac{1}{g} = \frac{1}{10} - \frac{1}{b}$$

as an unknown:

$$\text{solve}\left(\frac{1}{15} = \frac{1}{10} - \frac{1}{b}, b\right)$$

$$b = 30$$

as a generalizer:

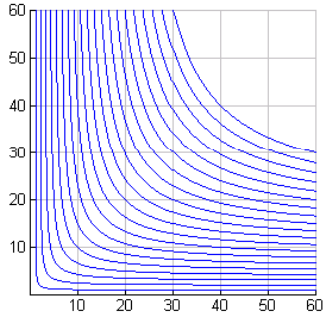
$$\text{solve}\left(\frac{1}{g} = \frac{1}{f} - \frac{1}{b}, b\right)$$

$$b = \frac{-f \cdot g}{f - g}$$

as a changing quantity:

$$b(g) := \frac{10 \cdot g}{g - 10}$$

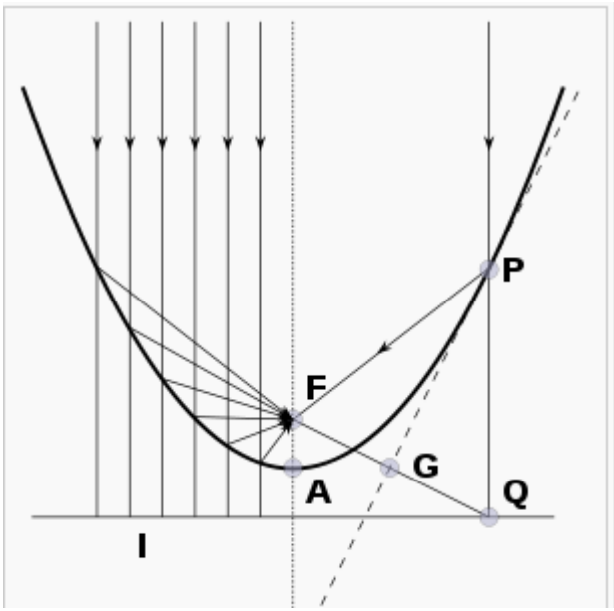
"Done"



From CAS to MRS

Multi-representational system

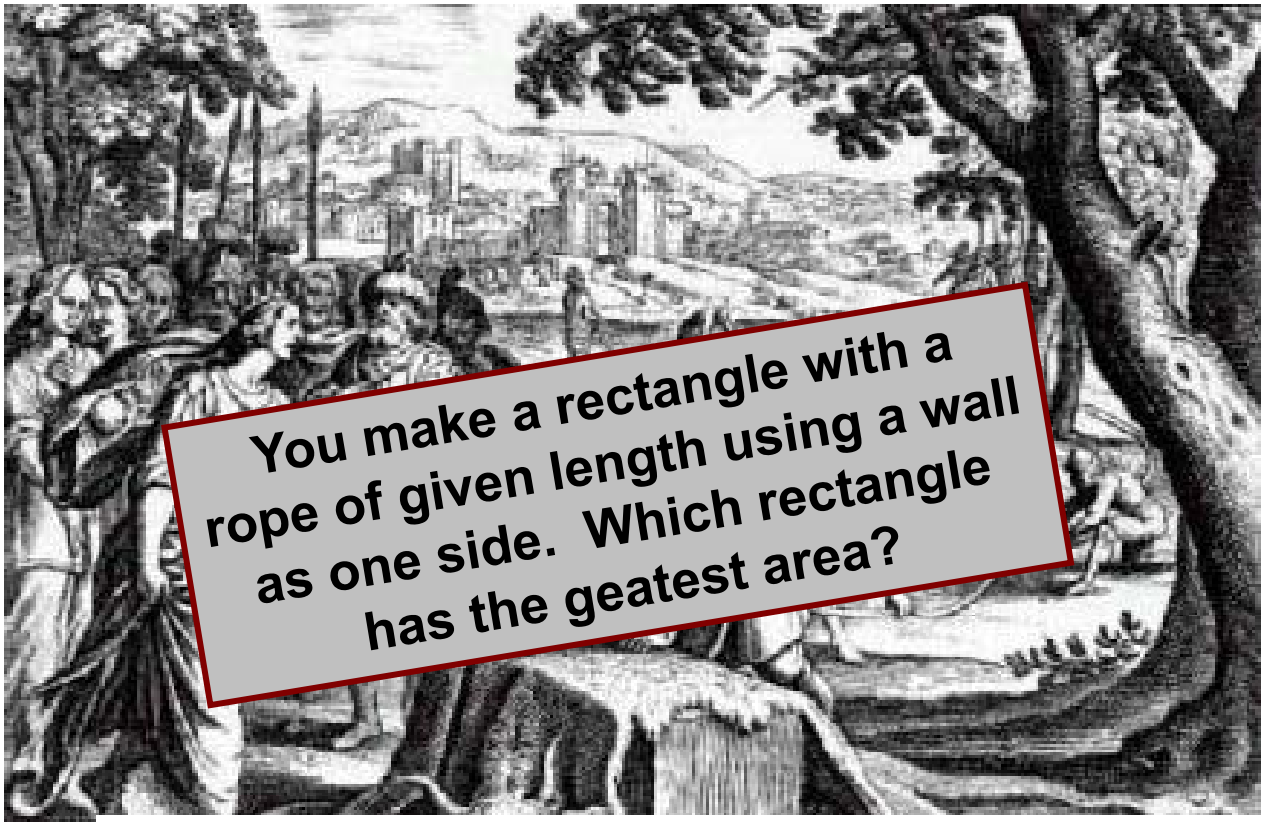
Why does a parabolic mirror work?



$f(x) = ax^2$

Focus point: $\left(0 \mid \frac{1}{4a}\right)$

From CAS to MRS



Devenere locos, ubi nunc ingentia cernes
moenia surgentemque novae Kathaginis arcem,
mercatique solum, facti de nomine Byrsam,
taurino quantum possent circumdare tergo (Aneis, Virgil)

Trying values

Spread sheet

Numerical approach

First step:

A value!!!

Geometrypackage

Find an equation

Computeralgebra

Symbolical approach

First step:

An expression!!!

u make a rectangle with a of given length usin one side. Which rec has the geatest are

Spread sheet

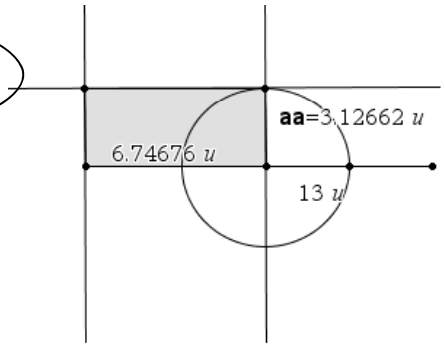
	A a	B b	C ar	D	E	F
◆	=seq(i,i,1,13,.25)	=13−2*a[]	=a[]*b[]			
1	1	11	11			
2	1.25	10.5	13.125			
3	1.5					
4						
		8.5	19.125			
	2.5	8.	20.			
8	2.75	7.5	20.625			
9	3.	7.	21.			
10	3.25	6.5	21.125			
11	3.5	6.	21.			
12	3.75	5.5	20.625			

A1 =1

Computeralgebra

$f(x):=(13-2\cdot x)\cdot x$	*Fertig*
$\frac{d}{dx}(f(x))$	$13-4\cdot x$
solve(13−4	$x=\frac{13}{4}$
solve(f(x)=0,x)	
solve(13−4·x=0,x)	25

Make a sketch



Graphical approach

First step:

A sketch/ a graph!!!

Geometrypackage

„The big 3 “ – relevant tools for learning mathematics

Spread sheet	Geometrypackage	Computeralgebra
Excel Access Apple Works	Cabri II Geolog ….	Derive MuPAd Maple Mathematica

GeoNext
GeoGebra

Just additive: V-200, TI-89, Classpad
Integrated & interactive on one surface: TI-Nspire (as Handheld & on PC)

„The big 3 “ – relevant tools for learning mathematics

Spread sheet | Geometrypackage | Computeralgebra

Research study in Norway (Fuglestad 2005)
3 grades (8-10, 14 – 16 years old pupils), 6 classes took part,
The three tools were always available

Which of the three tools do pupils use when solving tasks, where prinicipally all tools can support the solution?

All the three tools are used - by different pupils and the pupils could explain their choice according to their way of solution

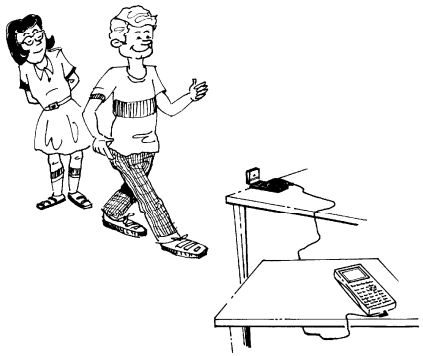
But : They must be aware of the three tools!!

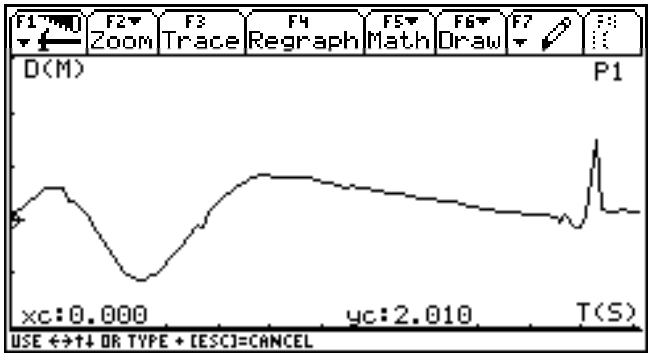
„The big 3 “ – relevant tools for learning mathematics

Spread sheet

Geometriepackage

Computeralgebra

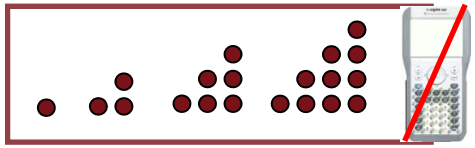




Algebra and functional thinking: How can it be realized?

Pupils age: 11 – 12 – 13 - 15

- An idea of a variable
- Multiple represen-tations
- Functional thinking
- An idea of an equation
- Types of function



Variables in formulas
x km · 0,04 l/km · 1, 56€

Sp Sh

Connections & Dependencies in geometry

GEO

Experiments to switch situation & graph

Gr T Sp Sh

The balance-model

CAS

Applied problems

CAS

Systems of linear equations

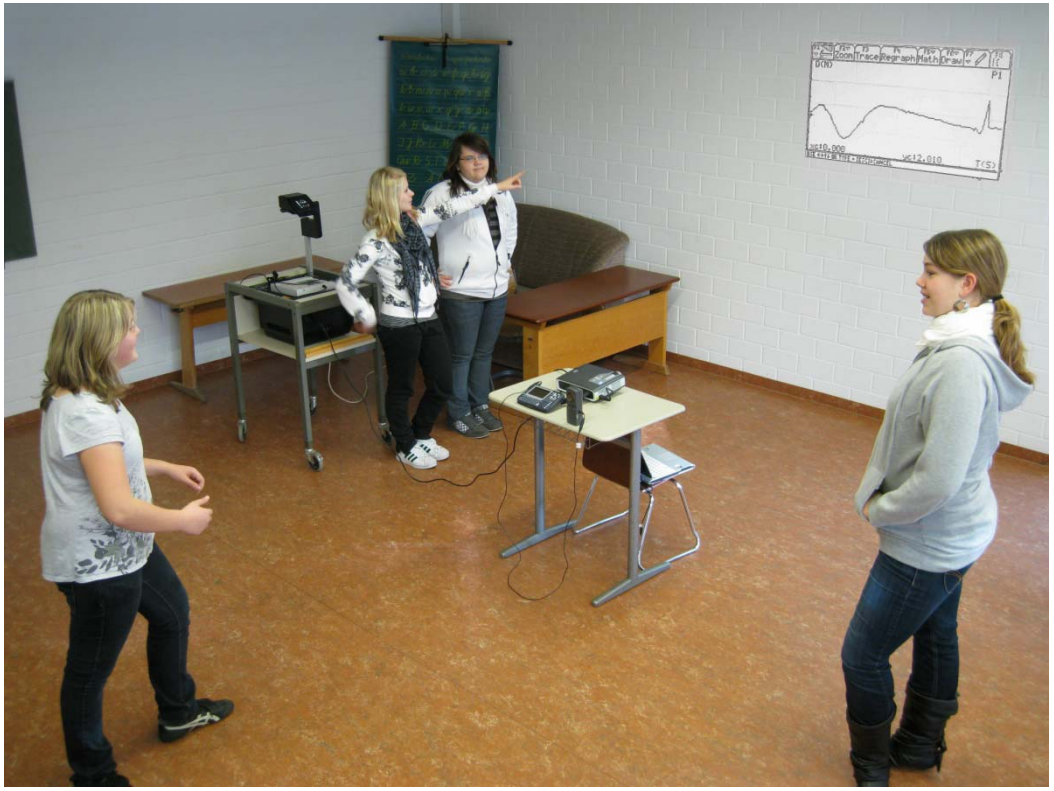
...

Introducing „funktional thinking“ with experiments

experimentieren im mathematischnaturwissenschaftlichen unterricht

Which graph is created by my movement?

Experiments to switch situation & graph

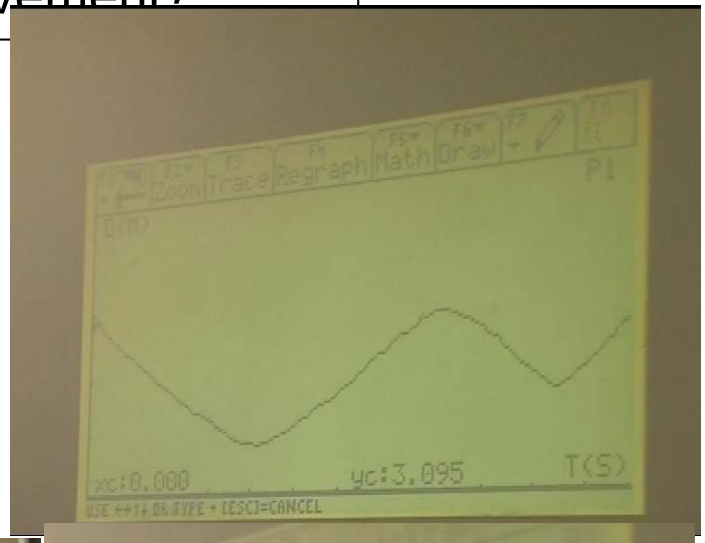


Research question:

How does an experiment supports the conceptualization of a „function“?

Introducing „funktional thinking with experiments

Which graph is created by my movement?

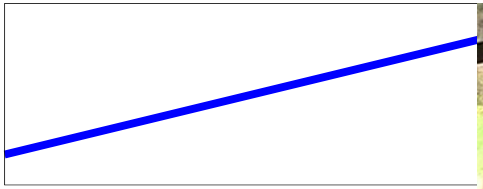


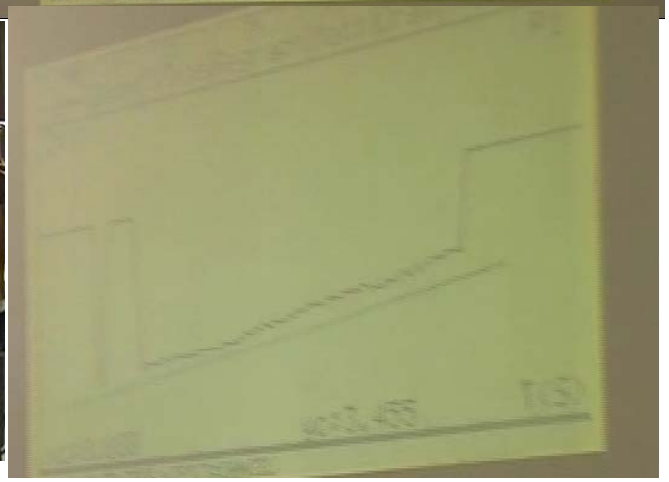
1. Step:

Spontaneous movements and interpreting the graph

2. Step:

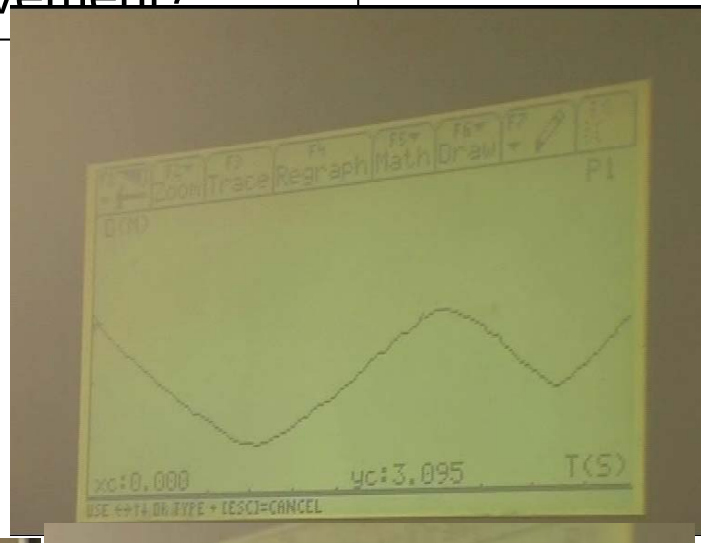
Match a given graph





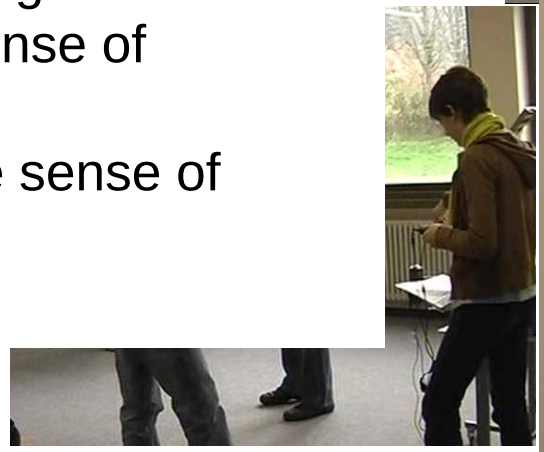
Introducing „funktional thinking with experiments

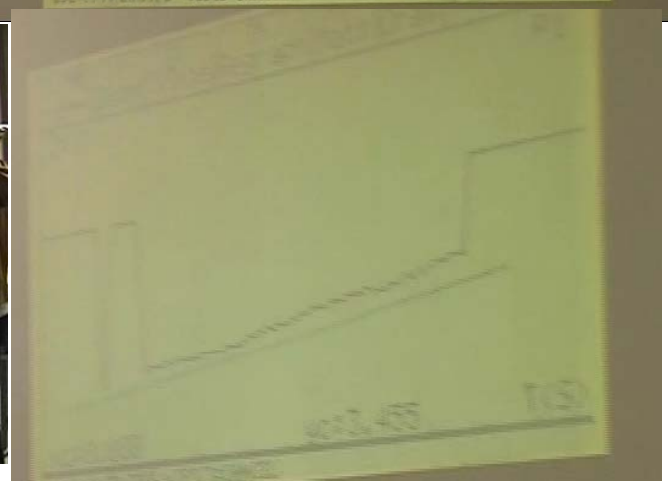
Which graph is created by my movement?



At first:
a very strong picture in mind as a pattern
„graph-as-a-picture irritation“

They overcome faults and argue in different ways, e.g.:
Statically: in the sense of mapping points
Dynamically: in the sense of covariation





Algebra and functional thinking: How can it be realized?

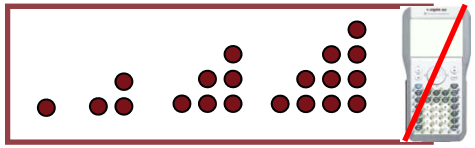
Pupils age

11

12

13 - 15

An idea of a variable



Multiple represen-tations

Functional thinking

An idea of an equation

Types of function

Systems of linear equations

...

Variables in formulas
x km · 0,04 l/km · 1, 56€

Sp Sh

Connections & Dependencies in geometry

G E O

Experiments to switch situation & graph

Gr T Sp Sh

The balance-model

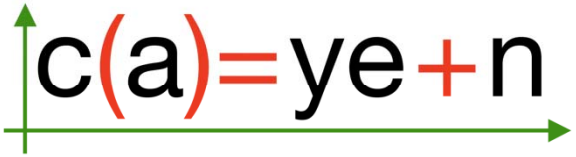
C A S

Applied problems

C A S

Using variables und functions

Applied problems CAS



CAS when learning algebra - Yes or No?

Bärbel Barzel, Mattias Zeller PH Freiburg

Paul Drijvers, Freudenthal Institute, Utrecht, Netherlands

Using variables und functions

Applied problems CAS

TI-nspire CAS versus TI-nspire

Using variables und functions

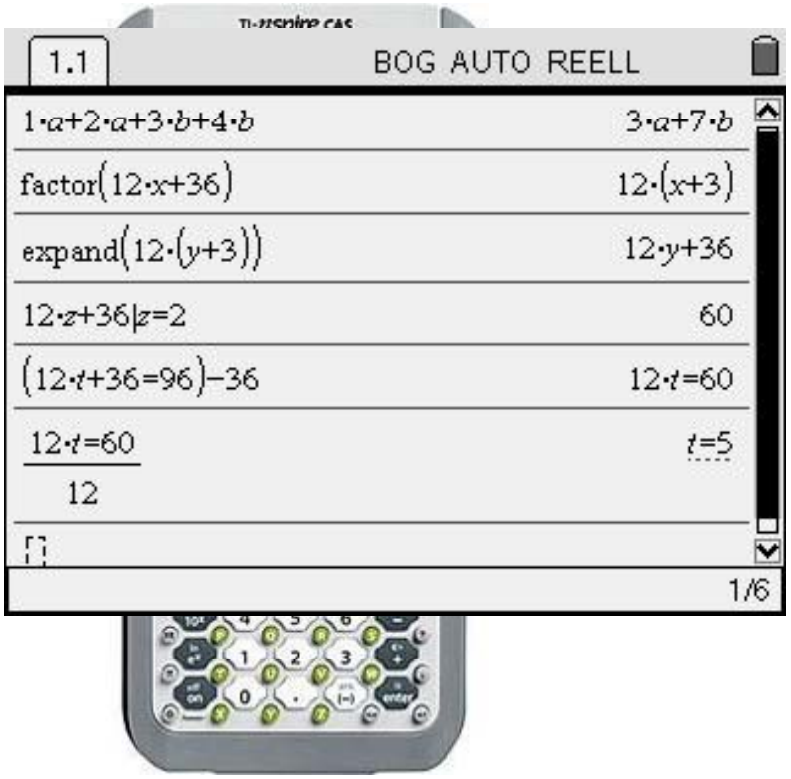
Applied problems CAS

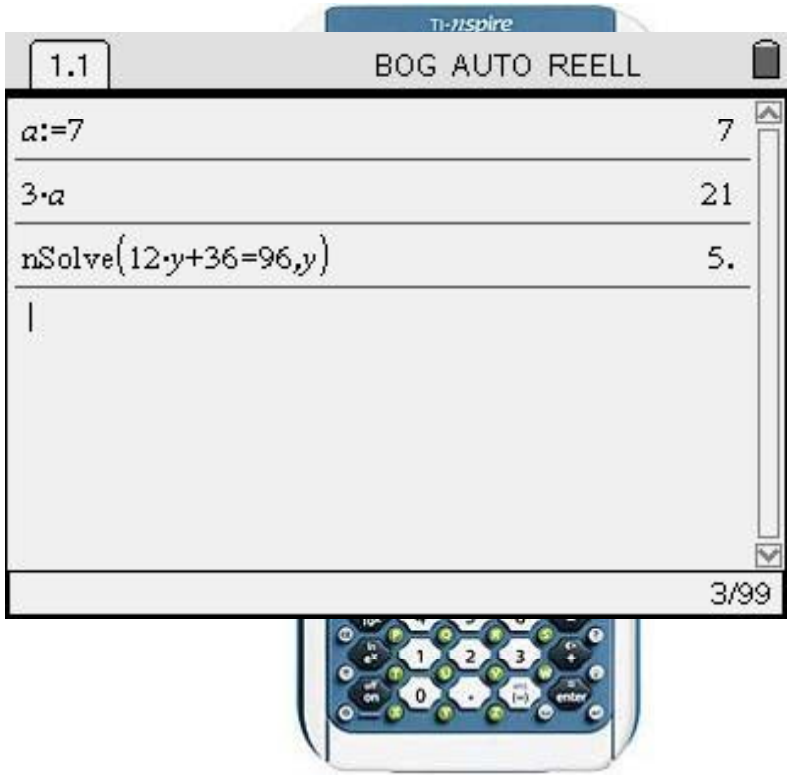
TI-nspire CAS versus TI-nspire

Using variables und functions

Applied problems CAS

TI-nspire CAS | versus | TI-nspire





Using variables und functions

Applied problems

Party organizers

After graduation, you want to organize a party. Several offers of local providers are available:

	YUKI EVENT	PARTYMAD	FLASH
meal / person	24 €	15 €	20 €
rent for the room	400 €	2300 €	900 €
stereo	350 €	400 €	included
decoration	included	200 €	300 €

Compare the offers.

These points might help you:

- How many guests do you expect? What will the costs be at which offer?
- When do two offers have an equal price?
- Present an overview of your results.

Additional task:

Another provider has a big party room. His offer is the cheapest, so long as more than 500 guests come. How much money can he charge, while still being the cheapest?

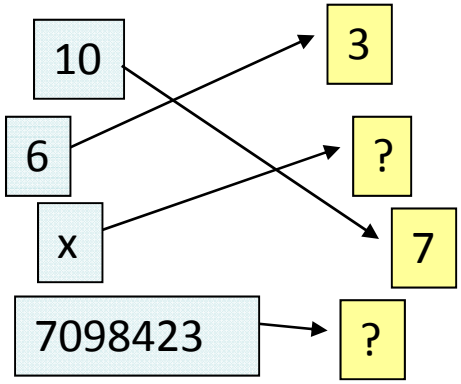
Using variables und functions

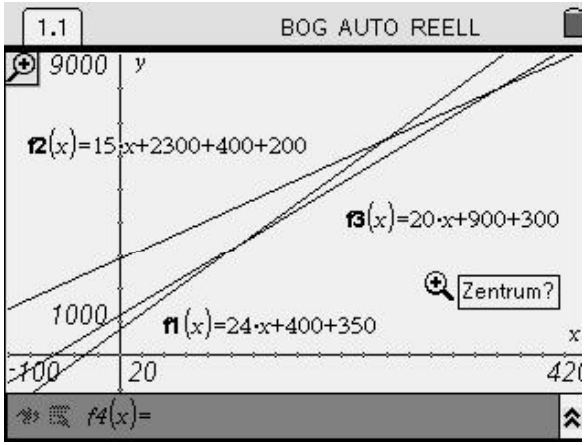
Applied problems

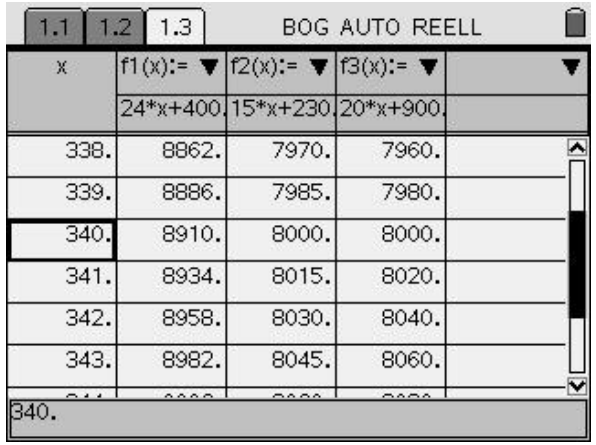
CAS

	Preis für...		in €
Gäste	Party	Event	Location
112	3438	4580	3440
113	3462	4595	3460
...	...	...	...
339	8886	7985	7980
340	8910	8000	8000
341	8934	8015	8020
	bis 112 Gäste	zwischen 113 und 340 Gäste	ab 341 Gäste

KLASSENFEIER







x	f1(x):= 24*x+400	f2(x):= 15*x+230	f3(x):= 20*x+900
338.	8862.	7970.	7960.
339.	8886.	7985.	7980.
340.	8910.	8000.	8000.
341.	8934.	8015.	8020.
342.	8958.	8030.	8040.
343.	8982.	8045.	8060.

340.

$$y = m\,x + b$$

... if I add 1 to the first number, than I have to ... to the other number ...

Using variables und functions

Applied problems C A S

Findings

1. The step from arithmetic to algebra:

- Non-CAS-pupils perceive a difference in the underlying rules of arithmetic and algebra.
- CAS-pupils accept results with variables as a result
  – Non-CAS-Pupils quite often do not.

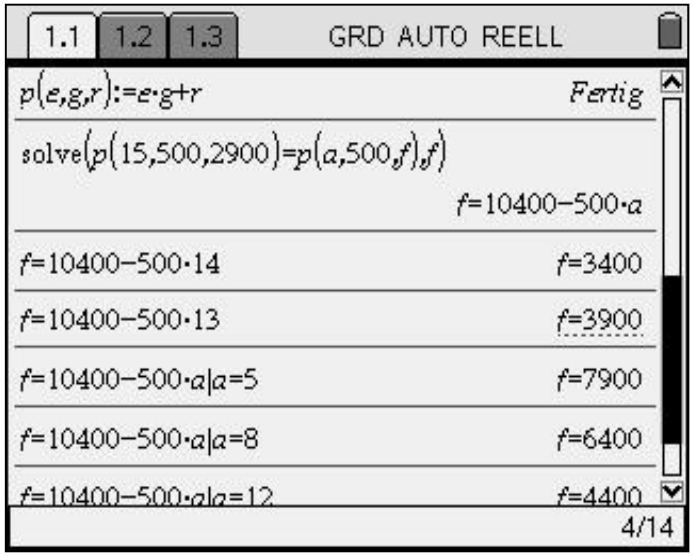
2. Using different representations

- Non-CAS-pupils avoid algebraic work,
  because it cannot be done by the machine.
- CAS-pupils are more motivated and keen to use variables,
  Non-CAS-pupils evaluate the algebraic representation as very difficult .

Using variables und functions

Applied problems

CAS



F

YUKI = 24·x + 750
↓ weiger ↓ mehr

FLASH = 20·x + 900
↓ weiger ↓ mehr

Party und = 15·x + 2900
das Neue Angebot weiger mehr

Wenn mehr Leute kommen wird das Essen billiger und die Miete mehr.

z.B. Neu: 14·x + 3000

Preis: p(e, g, r) = e·g + r
Essen Gäste Rest

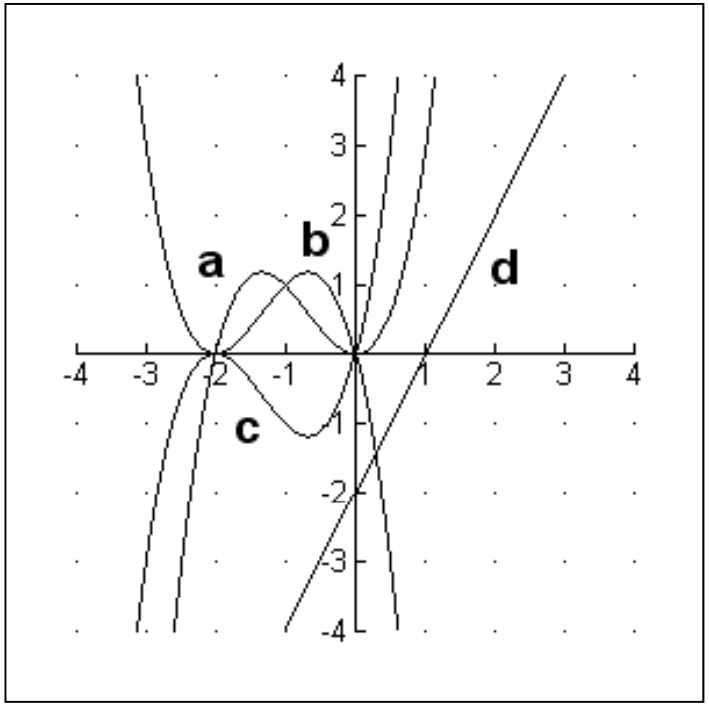
solve(p(15, 500, 2900) = p(a·500 + f), f)

f = 10 400 − 500·a
Rest neuer Abendessen pris

Das neue Angebot kostet z.B. 13 Essen und 3900 für den Rest.

Using variables und functions

Analysing pupils solutions



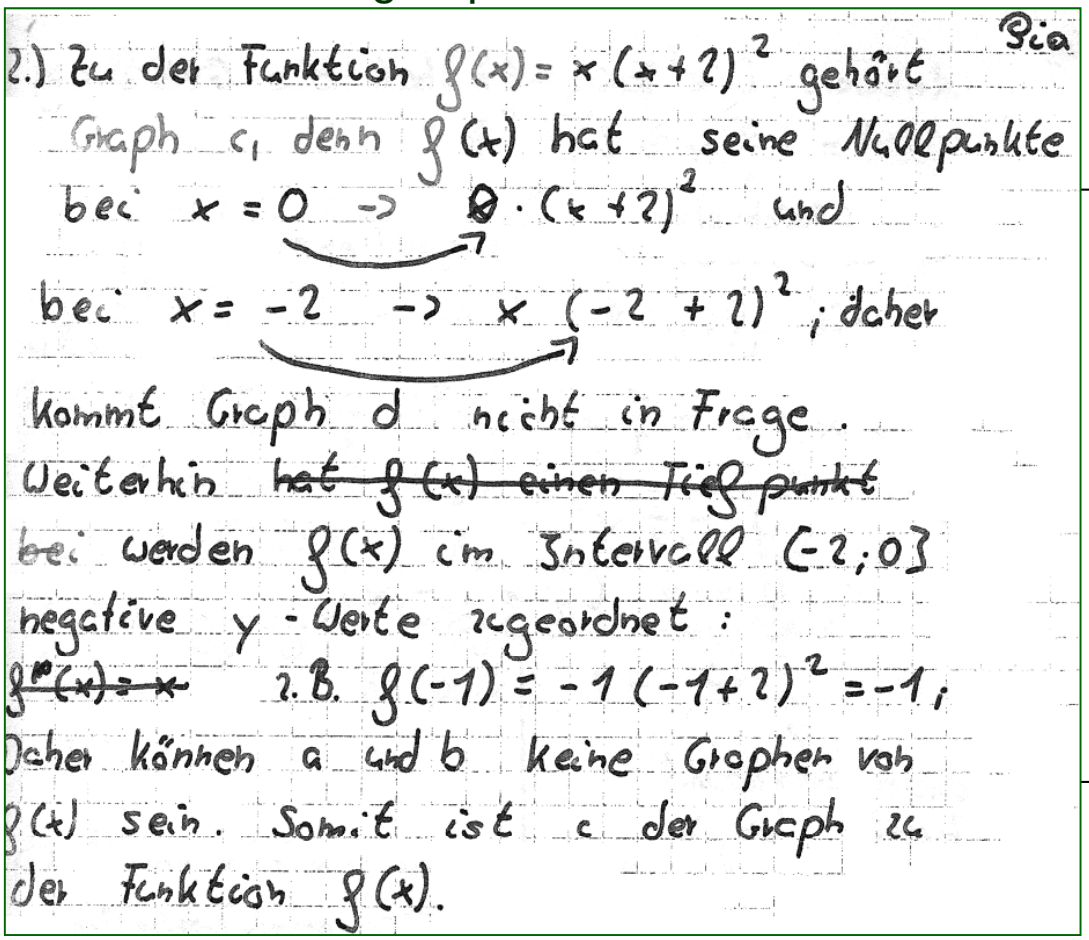
Which graph belongs to the function f with
$f(x) = x\,(x+2)^2$?
Give arguments!

CAS-group : 26
Comparison group without technology : 80

Using variables und functions

Analysing pupils solutions

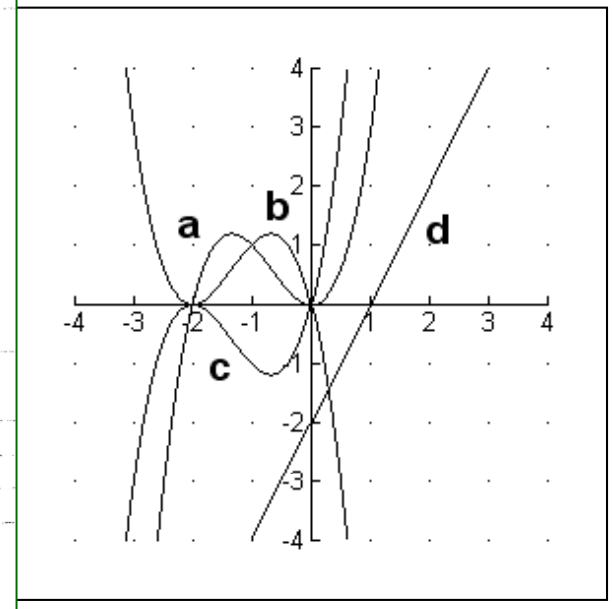
From the CAS-group:



Sia

2.) Zu der Funktion $f(x) = x(x+2)^2$ gehört Graph c, denn $f(x)$ hat seine Nullpunkte bei $x = 0 \rightarrow 0 \cdot (x+2)^2$ und bei $x = -2 \rightarrow x(-2+2)^2$; daher kommt Graph d nicht in Frage. Weiterhin ~~hat f(x) einen Tiefpunkt~~ ~~bei~~ werden $f(x)$ im Intervall $(-2;0]$ negative y-Werte zugeordnet: ~~$f''(x) = x$~~ z.B. $f(-1) = -1(-1+2)^2 = -1$; Daher können a und b keine Graphen von $f(x)$ sein. Somit ist c der Graph zu der Funktion $f(x)$.

$f(x) = x\,(x+2)^2$



Using variables und functions

Analysing pupils solutions

From the CAS-group:

Sia

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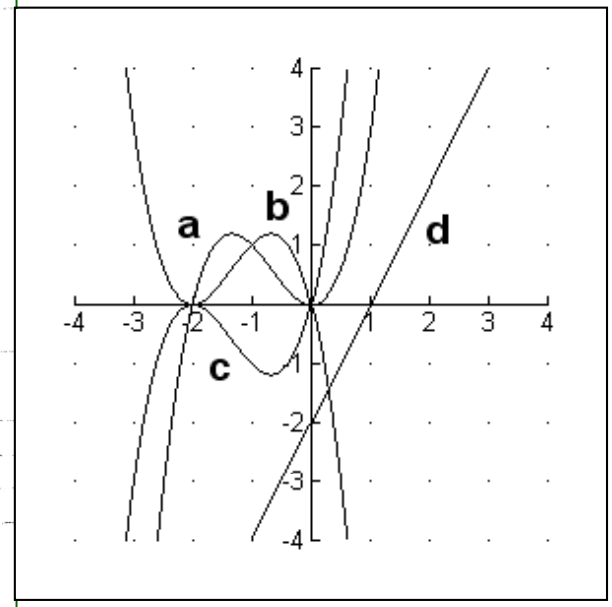
$f(x) = x\,(x+2)^2$



Using variables und functions

2) $f(x) = x(x+2)^2$

$f(x) = x(x^2+4x+4)$

$f(x) = x^3 + 4x^2 + 4x$

$f'(x) = 3x^2 + 8x + 4$

$f''(x) = 6x + 8$

$f'''(x) = 6$

$3x^2 + 8x + 4 = 0 \mid :3$

$x^2 + 2\frac{2}{3}x + 1\frac{1}{3} = 0$

$x_{1,2} = -1\frac{1}{3} \pm \sqrt{1\frac{7}{9} - 1\frac{1}{3}}$

$-1\frac{1}{3} \pm \frac{2}{3}$

$x_1 = -\frac{2}{3} \qquad x_2 = -2$

$f''(-\frac{2}{3}) = 4 > 0$ Min $\qquad$ Min $(-\frac{2}{3} \mid -2{,}07$

$f''(-2) = -4 < 0$ Max $\qquad$ Max $(-2 \mid 0)$

$f(x) =$

$f(-\frac{2}{3}) = (-\frac{2}{3})^3 + 4(-\frac{2}{3})^2 + 4 \cdot (-\frac{2}{3})$

$= -\frac{8}{27} + \frac{8}{9} - 2\frac{2}{3}$

$= -2{,}07$

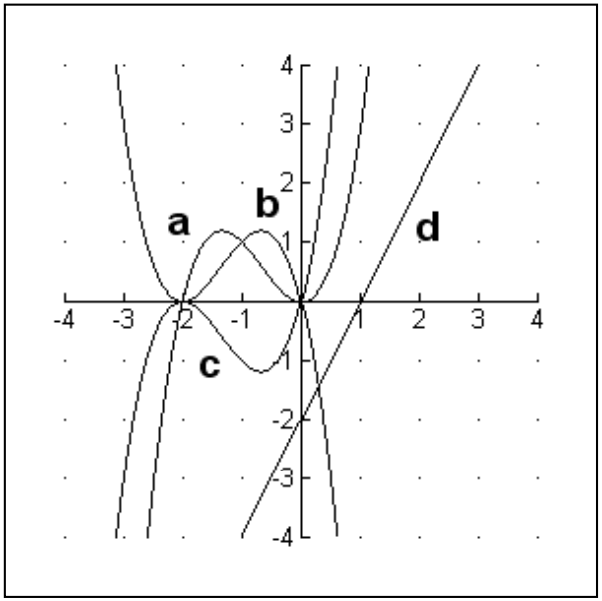
$f(-2) = 0$

Der Graph c gehört zur Funktion.
Da das Maximum mit dem Graphen übereinstimmt.

Analysing pupils solutions

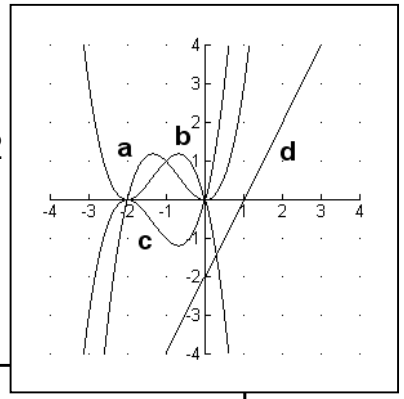
From the comparison group:

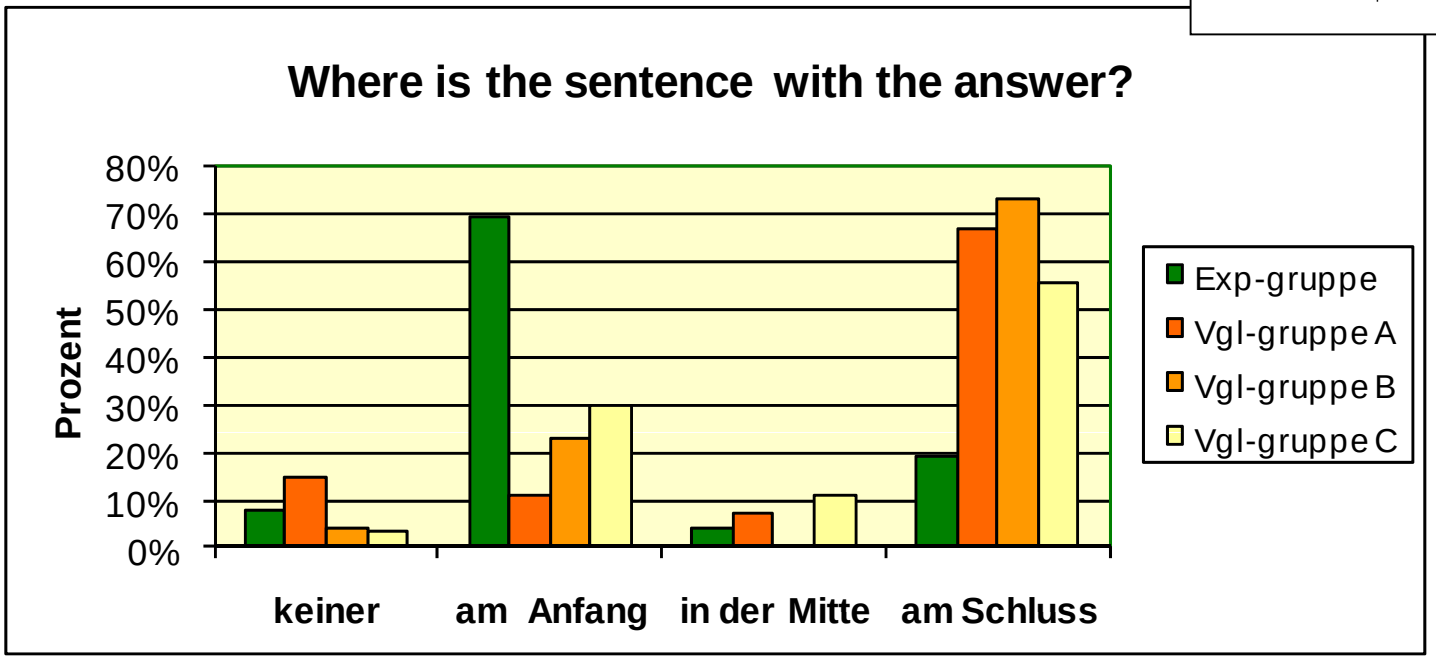
$f(x) = x\,(x+2)^2$



Using variables und functions

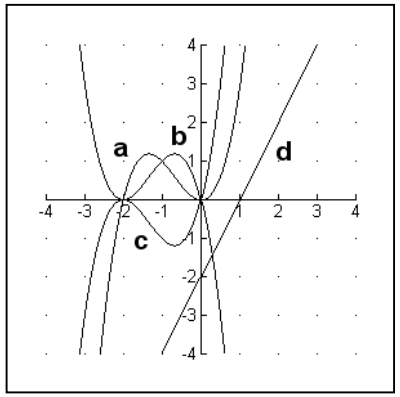
$f(x) = x\,(x+2)^2$



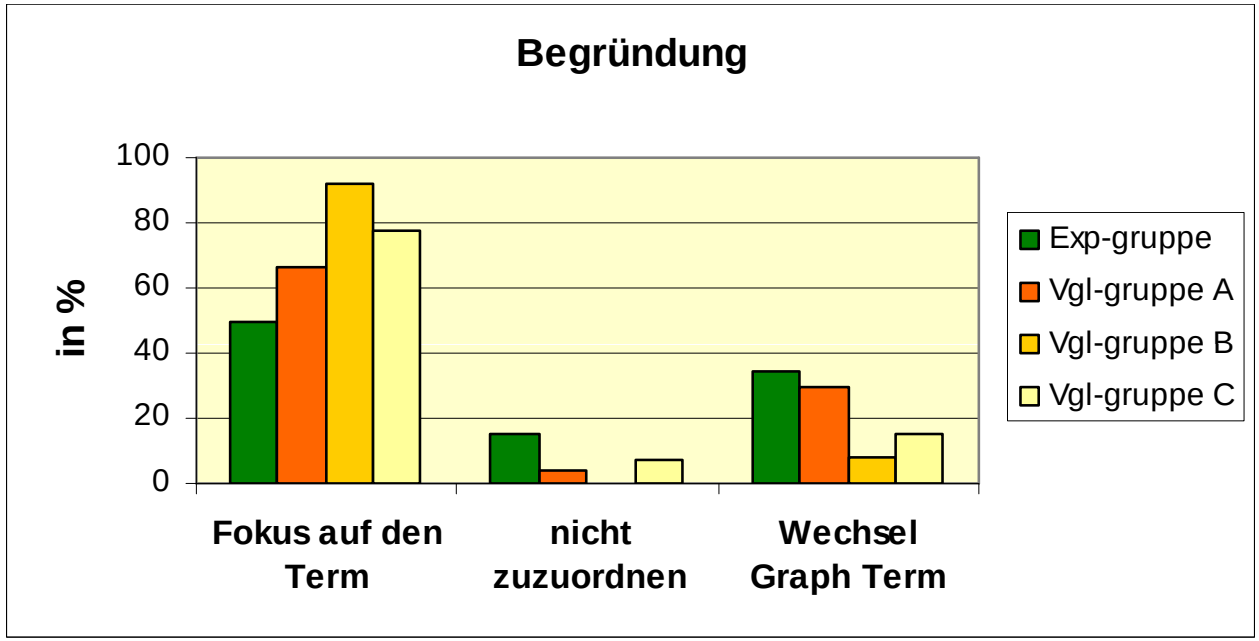


Using variables und functions

$f(x) = x\,(x+2)^2$

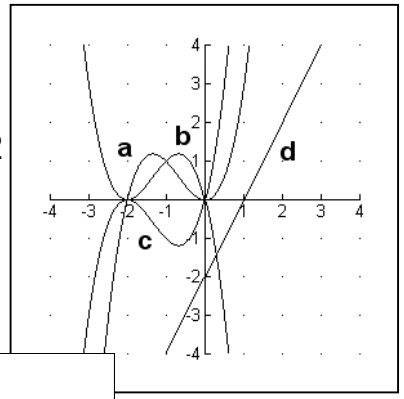


Possible explanations for : „sentence at the beginning!“



Using variables und functions

$f(x) = x\,(x+2)^2$



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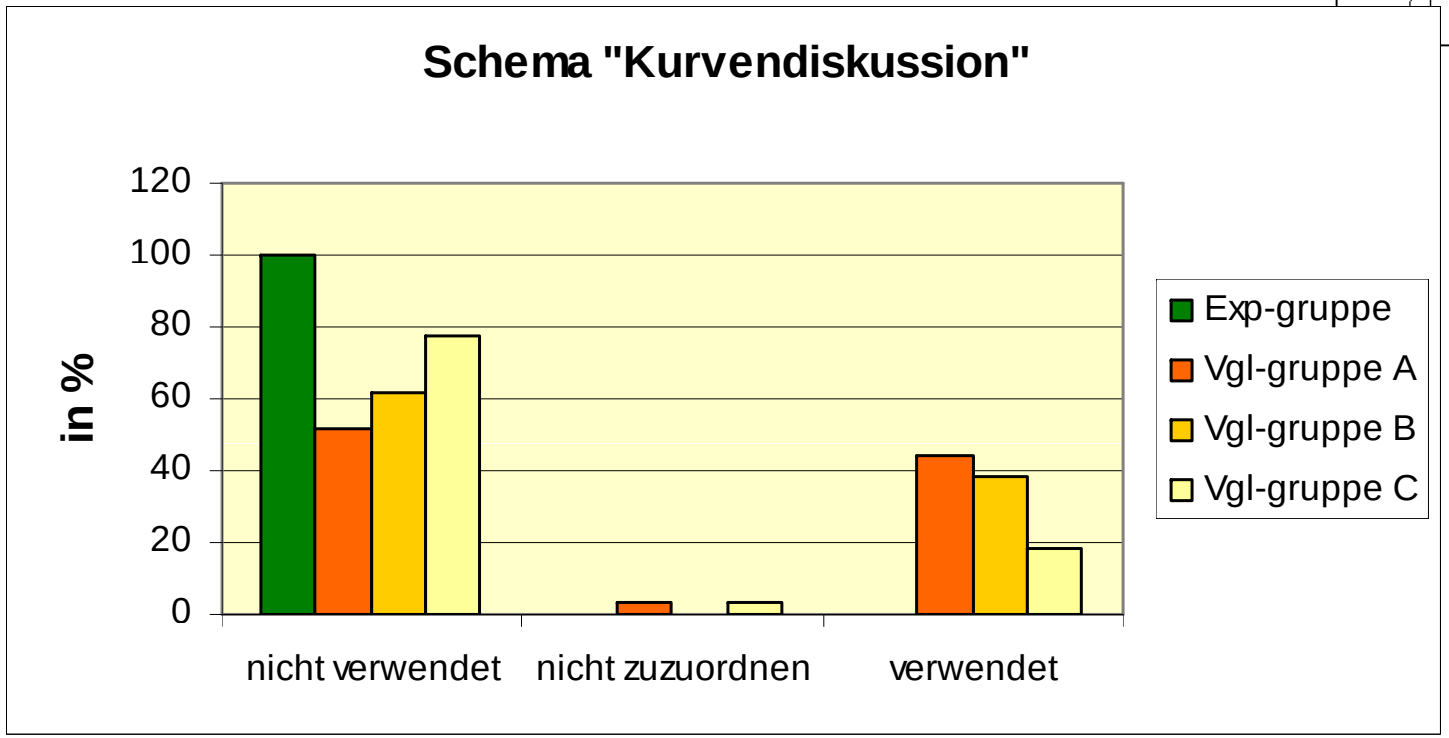


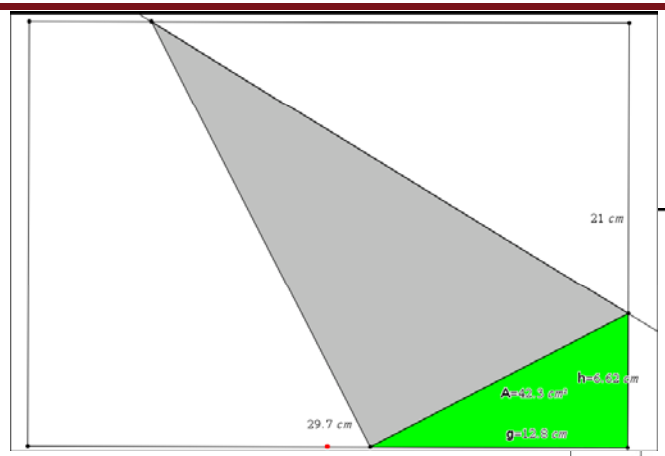
Using variables und functions

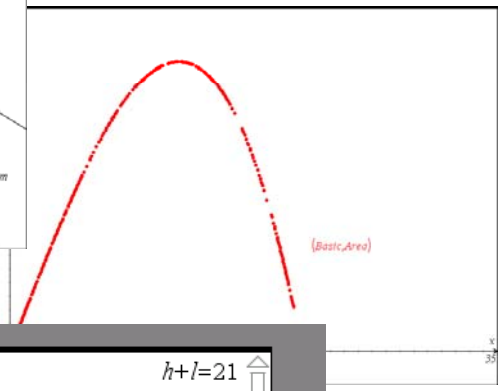
Applied problems

CAS

	basic	height	area
◆	=capture('	=capture('	=capture('
190	10.1499	8.04714	40.8388
191	10.3448	7.95204	41.131
192	10.4617	7.89411	41.2929
193	10.6566	7.79612	41.54
194	10.8125	7.71643	41.7169
195	10.8904	7.67615	41.7983
196	11.3582	7.42838	42.1864
197	11.4751	7.36481	42.256
198	11.592	7.30059	42.3143
199	11.709	7.23572	42.3614
200	11.7479	7.21395	42.3745
201	12.2936	6.9016	42.4228
202	12.4105	6.83282	42.3995
203	12.5275	6.76339	42.3641
204	12.6053	6.71682	42.3338
205	12.6443	6.69338	42.3165
206	12.6833	6.66988	42.2979
207	12.8389	6.57528	42.2098

F198



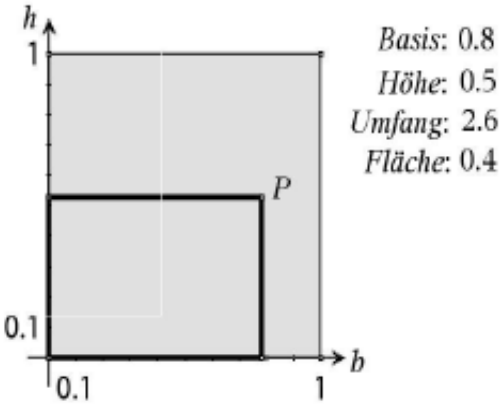


$h+l=21$	$h+l=21$
$l=21-h$	$l=21-h$
$g^2=l^2-h^2$	$g^2=l^2-h^2$
$(21-h)^2-h^2=g^2$	$441-42\cdot h=g^2$
$\sqrt{441-42\cdot h}\rightarrow g(h)$	*Fertig*
$\frac{g(h)\cdot h}{2}\rightarrow a(h)$	*Fertig*
$\frac{d}{dh}(a(h))$	$\frac{\sqrt{-21\cdot(2\cdot h-21)}}{2}-\frac{h\cdot\sqrt{21}}{2\cdot\sqrt{21-2\cdot h}}$
$\frac{d}{dh}(a(h))\rightarrow a1(h)$	*Fertig*
$\text{solve}(a1(h)=0,h)$	$h=7$

Using variables und functions

Applied problems

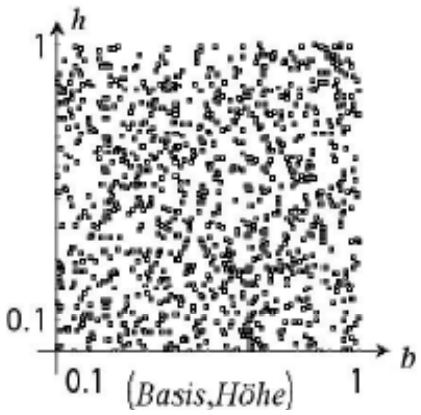
CAS

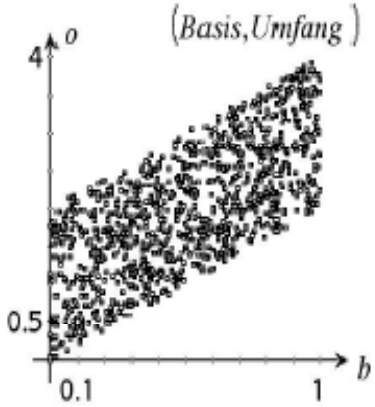


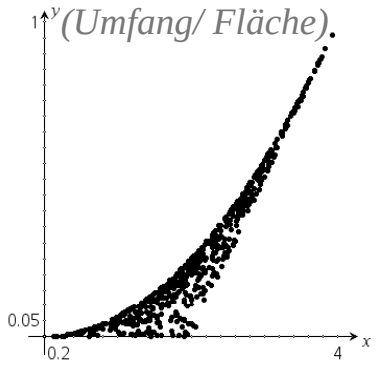
basis	B höhe	C umfang	D f...
◆1(999)	=rand(999	=2*'basis+2*'	
1647...	.418106...	2.05154220...	
2185...	.581118...	1.34927454...	
3971...	.882744...	2.67148418...	
4820...	.655161...	2.03028786...	
5019...	.848036...	2.46767698...	
6896...	.303560...	.707300000...	

C1 | 2.0515422031412

Felsager 2007







Supporting teachers to use ICT in the classroom

Aims:

- Convince them to use technology
  - By good examples
  - By experience and research results
- Learn them the handling of the technology
- Give them an idea of how to teach with it

Train teachers as an ongoing support,
not as a single event.

Supporting teachers to use ICT in the classroom

EdUmatics

European development for the use of mathematics technology in classrooms

2009-2012

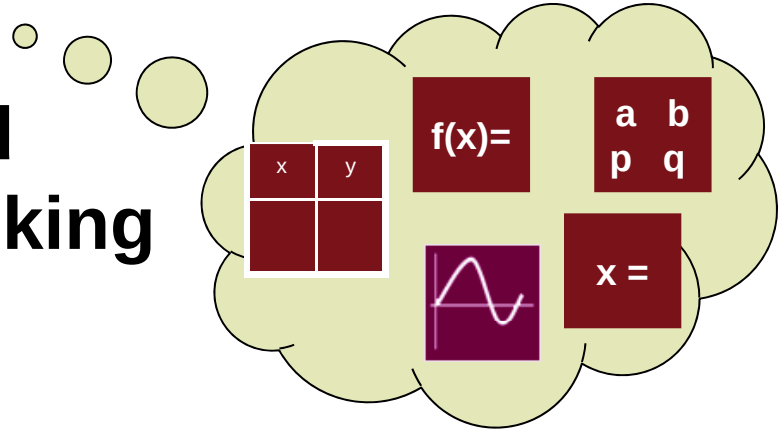
The proposed course will consist of 5 modules:

1. Starting to work with ICT;
2. From static to dynamic representations
3. Constructing functions and models;
4. Using ICT in the classroom: Teaching approaches;
5. Interrelationships between software.

Supporting teachers to use ICT in the classroom

(Module 1)	(Module 2)	(Module 3)	(Module 4)	(Module 5)
Starting to work with ICT	*From static to dynamic representations*	*Constructing functions and models*	*Methods of teaching with ICT*	*Interrelationships between CAS, GEO, Sp Sh*
University of Freiburg (DE) University of Maribor (SI)	**University of Lyon (FR)** University of Turin (IT)	**University of Paris (FR)** Charles University (CZ)	**University of Utrecht (NL)** University of Chichester (UK)	**University of Montpellier** (FR) University of Würzburg (DE)
Bärbel Barzel; Matija Lokar/ Vlasta Kokol-Volj/ Julijana Palcic	Luc Trouche/ Gilles Aldon; Ferdinando Arzarello/ Ornella Robutti	Michèle Artigue; Jarmila Novotna	Paul Drijvers; Alison Clark-Wilson	Caroline Bardini; Hans-Georg Weigand

The Role of CAS when learning algebra and developing functional thinking



Focusing

1. … the topic
2. … the technology
3. … the tasks & teaching
4. … the teacher training

Let´s do it manifold

CAS is not alone

Give chances & freedom

Support in an integrated way