Finding Holmes' Formula

Relating A Person's Stride Length To Their Height

SCENARIO

The night had been warm and very humid. Near closing time a small neighborhood market was robbed and the owner of the market killed. Police arrived on the scene quickly because an alarm had been sounded from the market. The suspect fled on foot through the rear door. At the rear of the store were a grassy area and a sidewalk. When police came through the rear of the store they found a clear set of wet footprints on the sidewalk. The footprints were immediately photographed and measured for size and length of stride. From this evidence alone they were able to determine approximate height, shoe size, any stride deviation and also had the ability to later match a suspect's shoe tread with the pattern left on the sidewalk.

BACKGROUND INFORMATION

It is generally believed that the first appearance of a genuine forensic investigator in literature was in the chronicles of the 60 Adventures of Sherlock Holmes by Sir Arthur Conan Doyle. Holmes with his sidekick, Dr. Watson, who was the alter ego of Doyle, was able to survey a crime scene and make remarkable deductions about the nature of the crime and the perpetrator(s) of the crime. All of Holmes' adventures, except for the last two, did not involve his "arch rival" the maniacal criminal minded Mathematics Professor, Moriarty. This characterization of a Mathematics Teacher is unfair and has scarred the reputation of Mathematics Teachers for years. However unjust this is, we need to get back to our subject.

In one of the early Adventures, A Study in Scarlet, Holmes makes the following observation in Chapter 15: "Why the height of a man, in nine cases out of ten, can be told from the length of his stride." He then, as he often did, asserts his mental superiority by saying, "It is a simple calculation enough, though there is no use my boring you with figures. I had this fellow's stride both on the clay outside and on the dust within." Holmes

does not choose to share with Watson, the investigating police officers, or us exactly how he made the calculation.

What formula did Holmes have in mind when he made his "simple enough calculation?" How much confidence can we have in this calculation? Holmes further mentions in Chapter 15 referenced above, "Then I had a way of checking my calculation. When a man writes on a wall, his instinct leads him to write about the level of his own eyes. Now that writing was just over six feet from the ground. It was child's play." Note the sarcasm at the end of the quote. Our mission in this Activity will be to attempt to discover Holmes' simple formula based on information we can collect using a measuring tape and Derive's® **Fit** Operator.

Your experimental method will be for you and your classmates take ten steps, measure the distance covered by those steps, and record the person's ID number, their height, the length of their step (note: this is actually a half of a stride), and, for a later investigation, their shoe size. Having collected the data, you will use the statistical list editor and the Linear Regression Tool to determine if Holmes' assertion is true and, if so, to approximate his formula for determining a person's height.

FORENSIC OBJECTIVES

- To investigate whether there is a relationship between the length of a person's stride and their height.
- To test a hypothesis that a person's shoe size is related to their height.

SCIENCE/MATHEMATICS OBJECTIVES

- To make two hypotheses about determining the height of an individual.
- To efficiently gather data to test these hypotheses.
- To use Linear Regression to build a mathematical model for the data.
- To determine the adequacy and limitations of the model.

MATERIALS

- Derive®
- A clear area at least 30 feet in length
- Chalk or tape to mark a starting line about 50 cm long for individuals to begin taking steps.
- Data sheets to record each person's ID Number, height, step length, and foot length.
- A meter stick

PROCEDURE

1. Your teacher will supply you with data sheets that contain the following columns

Individual ID #	Height (cm)	Step Length (cm)	Shoe Size

2. Set up three stations with a data collector and a data recorder at each station
   Station 1 – Measure and record each person's height in cm and record it together with their ID #.
   Station 2 – Measure and record each person's shoe size. Record the size together with the person's ID #.
   Station 3 – An area at least 10 meters long with a starting line. The data collector at this station will measure the distance covered by each individual in 10 steps. The recorder will divide this distance by 10 and record the average length for each step.

3. At Station 3 have each individual stand with their heels on the starting line and count out 10 normal steps. The individual needs to stop at the end of the tenth step and bring their heels together. Mark the position of the heels and measure the distance of the mark from the starting line.

4. When all of the data is collected merge it and record a complete record for each individual on a Master Data Sheet.

5. Create three Vectors; `Height, Step,` and `Size`, for the individual heights, step length, and shoe size, respectively. If you desire, you can also make a list of the ID numbers, although they will not be used in any of the calculations. Enter the data for each individual in the same location of each list, i.e. individual 3's data will be in the third position of each list.

6. Now you can begin the analysis. Start with graphing Height vs. Step Length. Begin by creating a Plot Screen aside of your Algebra Screen.

7. Set your `Plot Axis` variables for `Horizontal min` and `Horizontal max` to be a little less (no more than 5 cm) than the minimum step length and a little more than the maximum step length. For `Vertical min` and `Vertical max` do the same using the values for the heights. Use a `Scale` of 1 on each axis.

8. Graph the dot plot of the data you gathered by creating a matrix called HS defined by [Step, Height]` . Does there appear to be any organization to your data?

9. We will assume that the model that describes the relationship between height and step length is a linear one. To do this, use Derive's **Fit** function with *a*s+ b* as the target function.

EXTENSION ACTIVITIES AND PROCEDURES

EXTENSION #1

Recall that in the introduction, Holmes was quoted as saying that the calculation was "simple enough." The formula that your data inferred may have done a good job of approximating your data, but it probably involved some coefficients that were not easy to deal with. One would surmise that Holmes probably used a relation that involved integers or relatively easily handled fractions for the coefficient of stride length and the constant to be added. In this extension you will try to find a line with simple rational values for the coefficient and constant that may not be a close fit to your data, but is simple and 'close enough'.

1. Try using integers that are 'close' to the coefficient of step length in your relation and fill in the following table.

ID #	**x = Step Length**	**y = Height**	**(a_1 =) y – a_1x**	**(a_2 =) y – a_2x**

2. For the value of a_1 create a dot plot that has y – a_1x vs. Height. Can you find a constant line y = b such that most of the values for y – a_1x lie within a few centimeters of this line? If so, define the Y-variable Y2 to be a_1x – b and graph the line on your original dot plot of Height vs Step Length with the regression line.

3. Repeat Step 2 for the coefficient a_2 .

EXTENSION #2

This extension is almost exactly the same as the original Procedure given is steps 1 – 9 above. The only change that you will make is to use shoe length instead of height. Does the plot of the data show any apparent relationship?

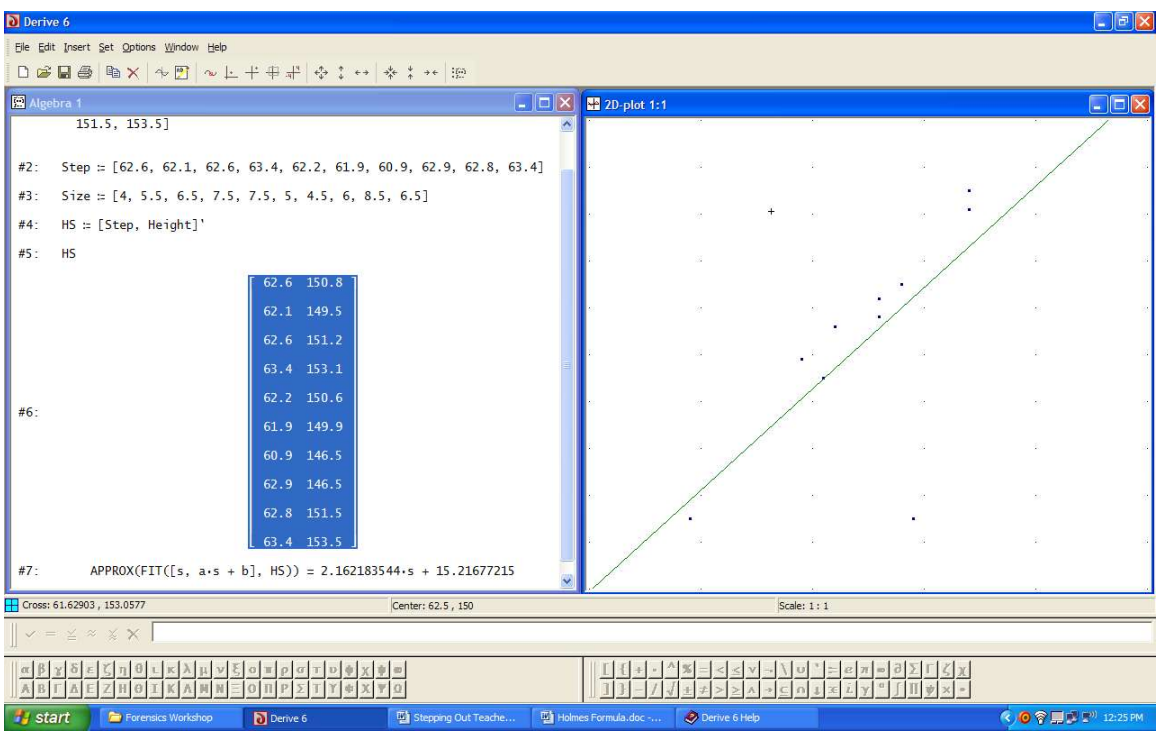
CORONER'S COMMENTS

The police immediately had a very good piece of evidence. Inside the store gave them the additional evidence of the weapon, possible finger prints, hairs inside the victim's hand and witness testimony. When all the evidence was collected, documented and analyzed they had a very clear picture of a possible suspect. A suspect matching this picture was then apprehended.

Some Sample Data

Data From a Hypothetical Class of Sixth Graders

Individual ID #	Height (cm)	Step Length (cm)	Shoe Size
1	150.8	62.6	4
2	149.5	62.1	5.5
3	151.2	62.6	6.5
4	153.1	63.4	7.5
5	150.6	62.2	7.5
6	149.9	61.9	5
7	146.5	60.9	4.5
8	146.5	62.9	6
9	151.5	62.8	8.5
10	153.5	63.4	6.5



WEATHER OR NOT

ESTIMATION OF TIME SINCE DEATH IN THE PRESENCE OF NON CONSTANT AMBIENT TEMERATURE

SCENARIO

It is 3AM and you are the Coroner on duty. You are called by the police to a wooded area by a stream. The body of a young man has been found by a fisherman who arrived early in order to claim a spot and start fishing at the first sign of dawn. Upon your arrival you note that the victim has suffered severe head trauma. The soil is not rocky and there appears to be no evidence of a blood trail leading up to the body. After making a preliminary survey for additional trauma and also preserving the scene of the incident, you take out your temperature probe and determine that the ambient temperature is 18.5^{o} C. You then make a small slit in the body slightly below the right portion of the rib cage and insert the probe into the liver of the victim. You record the temperature and then record temperatures every 15 minutes for an hour and a half. This is the data that you recorded.

Time	Liver Temperature ^{o}C
3:30 AM	34.00
3:45 AM	33.89
4:00 AM	33.67
4:15 AM	33.56
4:30 AM	33.33
4:45 AM	33.00
5:00 AM	32.78

The victim was a 19 year old, 96 kg male who was last seen at 9:00 PM the previous evening when he told a friend he was "going for a drive." The victim's car was subsequently found in the parking lot of a fast food restaurant approximately 5 miles from the incident scene.

BACKGROUND INFORMATION

Knowing the time of death is a very important factor in corroborating or breaking down a suspect's alibi. However, unless there is an eye witness to the incident, it is very difficult to determine the exact time of death. The use of temperature data can establish an approximate time of death, but even the best data and analysis can usually only determine the time to within an hour of the actual time. Bernard Knight, a renowned forensic pathologist from the University of Cardiff in Wales, states that the more exacting the pronouncement of time of death based on temperature data, the less it is to be trusted. The process of a body cooling is a complex process that depends on much more than the surrounding temperatures. On the other hand, if the inherent inaccuracy is accepted, a carefully done temperature based estimate can be useful in establishing a time interval during which the death occurred.

Temperature based estimates are generally based on *Newton's Law of Cooling*. This law states that for any given time, t, after the cooling process began,

$$r(t) = \alpha \cdot (T(t) - K) \qquad (1)$$

where $r(t)$ is the rate at which the body is cooling at time, t, $T(t)$ is the body temperature at that time, α, is a constant of proportionality depending on the body's characteristics, and K is the ambient temperature, assumed to be constant. This assumption results in a nice exponential form for the body temperature function, namely

$$T(t) = (T(0) - \mathrm{K})e^{\alpha t} + K \qquad (2)$$

This formula can be reproduced on our calculator by doing an exponential fit of $(T - K)$ vs t as suggested in the Derive® Help notes for the **Fit** operator. If we were to use this formula on the data in the scenario, we can make an estimate of the time of death assuming that the ambient temperature stayed constant from the time of death until the end of the sampling of the liver temperature. However, the victim was found out of doors and the ambient temperature was definitely not constant. It is not valid to use this simplified form of Newton's Law of Cooling to estimate the time since death.

If the ambient temperature is not constant, it is unlikely that a nice formula will exist for the body temperature function. In this case, it is necessary to approximate the values of the function, i.e. obtain a numerical approximation instead of a formula to find the values for the body temperature. In order to do this you will use the Cell Sheet application on your calculator and straight line interpolation to make the approximation.

The procedure for this activity is divided into two parts. For the first part you will measure the rate of cooling of a "body" when we have one large change in the ambient temperature. This will be done in a lab setting. The second part or Extended Procedure

will build on the first, but use actual meteorological data to determine a more realistic estimate for the time of death of the victim in the scenario. In both cases the cell sheet will be used to project temperature data back to the original or supposed original (37° C for the human body) temperature for the cooling body.

FORENSIC OBJECTIVE

- To obtain an approximate time when a death occurred to be used with other forensic evidence while investigating the scene of a death.

SCIENCE/MATHEMATICS OBJECTIVES

- To illustrate the role of *Newton's Law of Cooling* in making an estimate of the time of death.

- To use linear interpolation to estimate the function for the cooling of a body in the presence of a non constant ambient temperature.

MATERIALS

- None required except the data and Derive®.

PROCEDURE

Your local weather station is able to provide you with the following local weather temperature data for the date when the victim in the scenario was found.

Time	Temperature ^{o} C
5:00 AM	18.3
4:30 AM	18.4
4:00 AM	18.4
3:30 AM	18.5
3:00AM	18.5
2:30AM	18.9
2:00AM	18.9
1:30AM	19.2
1:00AM	19.3
12:30AM	20.0
12:00MN	20.4
11:30PM	21.1

11:00PM	21.7
10:30PM	22.2.

You are going to use this data to make an estimate of the time of death. We will approximate the data by a cubic cure. Thus, we will have a different one for each instant of time.

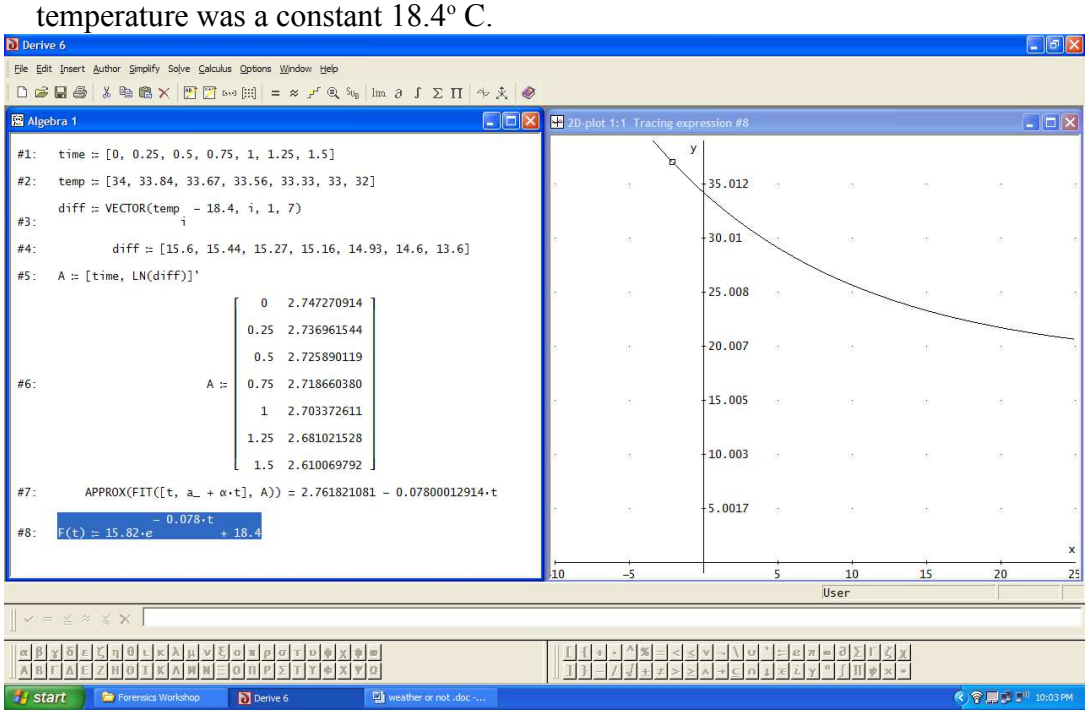
1. Create two vectors, `time` and `temp` using Derive's VECTOR command and the data in the scenario. The vector `time` should be the time since 3:30 AM in hours, i.e. [0, .25, .5, .75, 1.0, 1.25, 1.5].

2. The ambient temperature data between 3:30 AM and 5 AM seems to be fairly constant. Based on the data we will assume it to be 18.4° C during the time that the temperature data was extracted from the victim. Create a new vector called `diff` that contains the values in `temp` – 18.4.

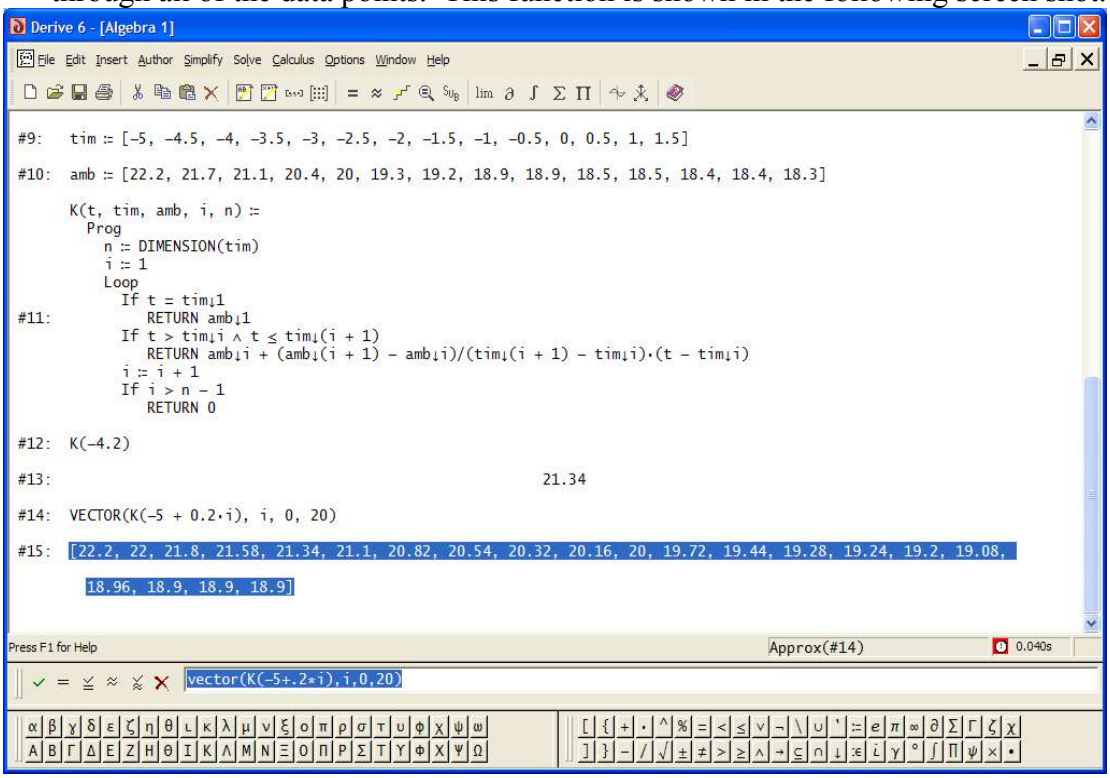
3. Our objective is to calculate the value for α in (1), Newton's Law of Cooling. To do this create a matrix `A:=[time,LN(diff)]`.

4. Write the expression `Approx(Fit([t,a_+αt],A))` and evaluate it. What is the value for α?

5. Create a function corresponding to equation (2) and graph it. Use the `Trace` operator to find the value for `t` when the body temperature is approximately 37° C. This is the estimate of the time of death (prior to 3:30 AM) if the ambient temperature was a constant 18.4° C.



6. We first need to get a function that represents the data for the ambient temperature. One possibility is to create a piecewise linear function that passes through all of the data points. This function is shown in the following screen shot.



7. Obviously, there is no nice analytical technique for solving equation (1) with the constant K being replaced by the piecewise linear function K(t). We need to find a numerical technique in Derive that will let us find the approximate time since death. NOTE: We are making the assumption that the death occurred after 10:30 PM.

8. Because it is the simplest of all of the numerical methods to explain that involve a rate of change, we will use Euler's Method. Use the Derive help pages for EULER_ODE to create a numerical solution for the body temperature function based on equation (1) with a variable ambient temperature. HINT: use a negative time increment.

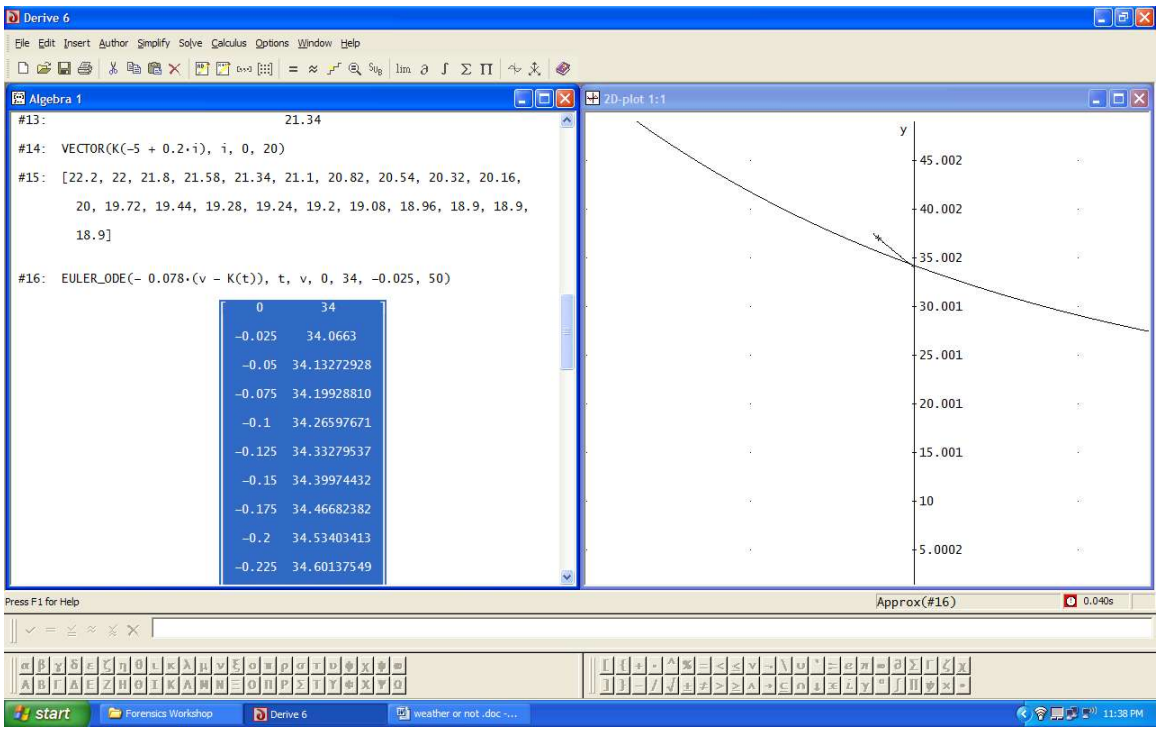
9. When do you estimate that the body temperature was 37° C?

CORONER'S COMMENTS

Although many TV shows seem to place a lot of emphasis on the estimation of the time of death, temperature data is one of the least reliable ways to make this estimate. Reports of when the victim was last seen give a good starting point. In the case of a death in the home, accumulated news papers, telephone records, evidence of the last meal eaten, and even buzzing alarm clocks may give a good indication. However, none are as generally

reliable as timely reports of when the victim was last scene. In the case discussed in the scenario, witnesses at a fast food restaurant said they saw the victim talking with three other "friends", and all four individuals got into the victim's car at about 10:30 PM and drove off. Subsequent, daylight investigation on the scene of the death revealed a tire iron in the high grass near the creek. Along the road near the scene, tire tread marks were found. A cast of these marks matched the treads of the tires on the victim's car. The three "friends" were later identified and interrogated. One of them confessed to a rather heated argument between the three and the victim. The suspect said that they took the victims car keys and drove off in his car leaving him to walk back to the restaurant. The three were eventually arrested and are awaiting trial.

A SCREEN SHOT OF THE SOLUTION



AN EXERCISE USING A GRAPHING CALULATOR AND TEMPERATURE PROBE

Collecting the Data

1. Firmly connect your temperature probe to your calculator. The EasyData Application will start and show the temperature that is being registered. This will ambient temperature of the room. It may take a minute or two until the probe settles down to a fixed temperature.

2. Prepare an "ice bath" by placing the ice cubes in the beaker (or bowl) and adding the water. Give the mixture a light stir.

3. Record the room temperature in your Lab Notes

4. Place the metal portion of the probe in "ice bath" and after the temperature data settles down, record the temperature in your Lab Notes. If possible, insulate or keep the bath cool to maintain a near constant temperature.

5. Choose `Setup` option `2: Time Graph ... Edit` to set the `Sample Interval` to `10` and the `Number of Samples` to `120`. This will result in a sampling time of 20 minutes or 1200 seconds.

6. While taking the temperature of the room and the ice bath, moisten the cloth with the warm (37° – 40° C) water.

7. Fold the cloth in quarters and wrap around the metal part of the probe. Place the entire probe in a plastic bag and seal the bag around the USB cable.

8. Set the timer to 10 min and `Start` EasyData gathering temperature data. Allow the probe to sit at room temperature for 10 minutes.

9. At the ten minute mark, immerse the bag with the probe in the ice bath. Be sure that the cloth is fully submerged, but that the water does not cover the top of the bag and seep into it.

10. When the calculator indicates the data collection is done, detach the probe from the calculator and remove it from the ice bath.

11. Quit the EasyData App. Save your data by placing the contents of L_1 in a list called `TIME` and L_2 in a list called `TEMP` (If your teacher is collecting your data, you may be given different names to use for these lists. We will use the names indicated here.)

12. In order to obtain the value of α for *Newton's Law of Cooling* (Equation 1), follow these steps. We will need the value of α to reconstruct the Temperature Function for the wash cloth.
    a. Extract the first 600 seconds of the cooling process (room temperature) from L_1 and L_2 with these steps:
       `seq(L1(I),I,1,61) STO L1`
       `seq(L2(I),I,1,61) STO L2`
       During this time period of our experiment the ambient temperature was constant so, the graph of the data should match equation (2) above.
    b. Use an exponential regression of this list (NOTE: during this time the ambient temperature was a constant.)
       `ExpReg L1,L2`
       Copy the value for b in your Lab Notes
       (b is actually the value for e^{α} in equation (2) above)
       `LN(`your value for b`) STO A`
       The value stored in A is the value of α in equation (1) above. Enter the value for A in your Lab Notes

13. (Optional) Check this value for α by storing values 61 to 121 of `LTIME` in L_1,and values 61 to 121 of `LTEMP` in L_2 repeating `ExpReg L1,L2`. The value for b should be close to the one obtained for the first 61 data points. It will not be exactly the same because the ice bath is experiencing some warming and, thus, having a small effect on the data. This will affect the value of α.

Setting Up a Cell Sheet to Approximate the Data

1. Save the list `LTIME` in L_1 (`LTIME STO L1`). Reverse L_1 using the command `SortD(L1)`. This works since L_1 is in ascending order.

2. Save the list `LTEMP` in L_2 (`LTEMP STO L2`). Reverse L_2 using the command `SortA(L2)`. This also works because L_2 is in descending order.

3. Open the `CellSheet APP` and under `Menu` choose option `1: File` and then `3: New`. Give your cell sheet a short name as directed by your teacher.

4. For the moment turn off the automatic calculation feature of the `CellSheet` by choosing option `1: File` of `Menu` and then option `5: FORMAT`. Set `AutoCalc:` to `N` and leave all other settings as they are.

5. Set up the sheet as follows:

Cell	Value to type (you highlight the cell and press `ENTER`)	
A1	"A =	(in the Cell Sheet, = is obtained by `STO`)
B1	=A	(the value you stored in step 12 above)
C1	"AMB1	
D1	<the room temperature>	
C2	"AMB2	
D2	<the temperature of the ice bath>	

```
A3 & A4     "TIME & "(SEC)
B3 & B4     "ACTUAL & "TEMP
C3 & C4     "AMB & "TEMP
D3 & D4      "CALC &  "TEMP
```

6. Move the cursor to cell A5 and choose option 3: Options under Menu. Now choose 4: Import/Export and then 1: Import List. For List Name: type L_1 and leave all other information the same (assuming that the 1st Cell: is listed as A5.

7. Move the cursor to cell B5 and repeat step 6 except that column B will be filled with list L_2.

8. Now move the cursor to cell C5 and will fill half of this column with the temperature of the ice bath that is stored in cell D2. Choose option 3: Options of Menu and then option 2: Fill Range with the following values:
   Range: C5:C64
   Formula: = D2 (= is STO, $ is 2ND STO)
   This will fill the cells C5 to C64 with the value in cell D2. The $ indicates an absolute address instead of a relative address that changes for each new cell to a cell in the same relative position to the new cell as cell D2, i.e. 3 up and 1 right.

9. Place the value of the room temperature in cells C65 thru C125 by repeating the same choices in Menu except this time you type
   Range: C65:C125
   Formula: = D1

10. Highlight cell D5 and type in the value that appears in cell B5. You are going to start at this temperature and see if you can work you way back, using linear interpolation, to the temperature that you started with when you first started collecting data. This is comparable to starting with the temperature of the body when you first arrived on the scene and trying to determine when the temperature of the body was last 37° C (normal body temperature). This would be a way to approximate the time of death.

11. Now it is time to turn AutoCalc back on. Refer to step 4 except that you now set Auto Calc to Y.

12. Making the assumption that the change in the temperature is linear over a relatively short interval from time, t, to time $t + \Delta$, we assume that the temperature at time, $t + \Delta$, is: the temperature at time $t + \Delta \cdot$ rate of change in temperature. In this case $\Delta = 10$ sec and by *Newtin's Law of Cooling*, the rate of change in temperature is: $\alpha \cdot$(temperature at time t – ambient temperature). So, in cell D6 type:
   = D5 + 10*B1*(D5 - C5)
   You will be copying this formula for the rest of column D.

13. After the CellSheet finishes recalculating (this will take some time), highlight cell `D6`. Press `F1` and choose `Copy`. Next choose option `2: Edit` and then option `4: Select Range` and type

    `Range: D5:D125`

14. After pressing `Enter` and returning to the spread sheet, choose `Paste`. The CellSheet will recalculate for a while and finally the calculated values will appear.

15. Choose Option `2: Edit` in `Menu` and then choice `1: Go to Cell …` and type `D125`. This will take you to the beginning of your data collection. Compare this value to the value in cell `B125`. How do they compare? Enter the two values in your Lab notes.

16. To better understand the approximation process used in this activity, choose option `4: Charts` from the `Menu` and then choose option `3: Line`. Type the following information in the resulting window

    ```
    XRange: A5:A125
    YRange1: B5:B125
    YRange2: D5:D125
    Title: TEMP DATA
    ```

    and then choose `Draw Fit`. This will draw the graph of the original data and then the graph of the calculated data. The two graphs are not the same, but note that the calculated data has the same general shape as the original data. If your data collection was fairly accurate, the two graphs are not too far apart. NOTE: Interpolation is not the best method of approximation, but it is the basis for other, more accurate methods

ALPHABET SOUP

DECODING A VICTIM'S MESSAGE IN CONNECTION WITH A DEATH SCENE INVESTIGATION

SCENARIO

You are called to assist in an investigation of a suspected suicide at a house in the upscale neighborhood in your town. You are met at the door by the police investigating the scene, who are questioning the victim's husband. You proceed to the office on the first floor and find the victim, a woman in her forties, slumped over her desk. It is apparent that the scene has been tampered with since the top of the desk is cleared except for a drinking glass that is half full. As you are documenting the crime scene with photos and case notes, you find a top desk drawer that apparently has had materials hastily thrown into them. You suspect that it contains items that had been on the desk top. Among them are an empty container of barbiturates and also a rather strange note that is shown below. After conferring with the police you find that the husband, who is dependent upon his wife's income, is mainly concerned with having the death declared an accidental death and not a suicide, stating that his wife often took sleeping pills before bed time and must have accidentally taken too many. This will enable the will to be probated quickly allowing him to gain his inheritance. After examining the label on the barbiturate container, you suspect that given the date the prescription was filled and the number of pills prescribed, the victim had taken a number that was far beyond what someone could have taken by accident. In addition, you are curious about the "note" found in the desk drawer that contains a ten digit number at the bottom.

dv kadautfwa dra zhll xvka dv tc ehwi hxxvgwd gla dra wgteau eamvj. zhfu dra kfqfdl. nvu ahxr zhfu xgea dra wgteau hwk draw dhia dra uathfwkau jraw kfbfkak ec nfndaaw. zmhxa dra ualgmdl dvqadrau hwk cvg rhba fd!

1113051208

Figure1: The Curious Note Found At The Death Scene

BACKGROUND INFORMATION

The message is obviously given in some form of a code. One might suspect that the number at the bottom is of paramount importance to the case. You might be sure that is the case if you have read a recent best seller that involving Swiss Banks and lock boxes containing clues. You are fascinated by the fact that the nonzero digits of the number are all taken from the first six of the so called Fibonacci Numbers: 1, 1, 2, 3, 5, 8. Further investigation by the police indicates that the victim's bank has explicit orders to release the funds in her rather large "endowment" account to individuals who know the pass number of the account. The number at the bottom of the note is not that number. The case investigators decide that it is important to learn the contents of the note.

Codes are almost as old as written communication. Certain activities required secret information or, at least, were perceived to need secrecy. One of the oldest codes was the so-called "Caesar Cipher" attributed to Julius Caesar for communicating with his generals. The idea is simple, choose a number, say 5, and transpose each letter in the alphabet by that number. For example A becomes F, B becomes G, etc. At the end of the alphabet, the letters will 'wrap around', i.e. X becomes C. This was effective for a while, but code breakers soon realized that they could recover the message in at most 25 tries. In fact, if they took one fairly long coded word and found a number that transformed that code into a known word, they very likely had the number to decode the entire message.

The logical next step in the progression of creating codes was the "Substitution Cipher." The person doing the encoding simply 'scrambles' the alphabet and then substitutes the letters in the scrambled alphabet for the letter in the same position in the actual alphabet. For example, if R is the first letter in the scrambled alphabet, it is substituted for each A in the message, etc. The person receiving the message also has a copy of the scrambled alphabet, or 'key', and reverses the process. Unfortunately, this scheme, although popular for many newspaper puzzles, is only that, a puzzle. It does not yield a secure coding method. The frequency at which letters appear in any language is well known and any relatively long message will exhibit this frequency. Figure 3 shows the frequency of letters of the alphabet for the English Language. Figure 2 shows letter groupings where any one letter in the group may precede other letters in the same group within a message.

I	E
II	T
III	A, O, N, R, I, S
IV	H
V	D, L, F, C, M, U
VI	G, Y, P, W, B
VII	V, K, X, J, Q, Z

Figure 2: Letter groupings for the frequency of letters in the English Language

The following table and text are from:
Fletcher Pratt, *Secret and Urgent: the Story of Codes and Ciphers* Blue Ribbon Books, 1939, p. 252.

Rank	Letter	Frequency of occurrence in 1000 words	Frequency of occurrence in 1000 letters
1	E	591	131.05
2	T	473	104.68
3	A	368	81.51
4	O	360	79.95
5	N	320	70.98
6	R	308	68.32
7	I	286	63.45
8	S	275	61.01
9	H	237	52.59
10	D	171	37.88
11	L	153	33.89
12	F	132	29.24
13	C	124	27.58
14	M	114	25.36
15	U	111	24.59
16	G	90	19.94
17	Y	89	19.82
18	P	89	19.82
19	W	68	15.39
20	B	65	14.40
21	V	41	9.19
22	K	19	4.20
23	X	7	1.66
24	J	6	1.32
25	Q	5	1.21
26	Z	3	.77

"The average length of English words is 4.5 letters per word.

Figure 3: Frequency of letters of the alphabet in the English Language

Figures 2 and 3 are taken from the web pages published by the Santa Cruz Public Libraries and are located by the URL:

http://www.santacruzpl.org/readyref/files/g-l/ltfrqeng.shtml

These web pages also have the frequency of letters for other languages such as German, French and Spanish. They can be accessed by placing *ger*, *fr*, or *sp* in place of *eng* in the part of the URL that reads ltfrqeng.shtml.

The frequency charts proved to be the demise of the Substitution Cipher as a secure method of coding important messages. While it is unlikely that any short message will adhere to these frequencies exactly, for longer messages the ordering of the frequencies of the letters in the message generally follows at least the ordering of the frequencies of the most common letters. If these letters can be identified it is very likely that the message can be deciphered. However, shorter messages can violate the expected letter frequencies. For these messages Figure 2 can provide a starting point.

Other coding schemes replaced the substitution cipher for secure communication of military and government documents. One invented by the French Cryptographer, Blaise de Vigenere used a key document where letters in the document to be coded were shifted by the value of the corresponding letter in the key. The coding and decoding of a message with this scheme is absolute simplicity. Just line up the letters of the document to be coded with the key and shift the document letter by the positional value of the key document letter. For example suppose the first word of the message is "see" and the first letter of the key is "this". Then the coding goes as follows:

s -> letter at (position of "s" + position of "t") Mod 26 +1 = letter at position 14 = *n*
e -> letter at (position of "e" + position of "h") Mod 26 +1 = *m*
e -> letter at (position of "e" + position of "i") Mod 26 +1 = *n*

So the coded version of "see" is "nmn". If the key is shorter than the message, then continue by starting over again at the beginning of the key.

To decode, simply repeat the process with the modification of subtracting the position of the letter in the key from the corresponding letter in the coded message and subtracting 1 instead of adding 1 to find the position of the decoded letter. Decoding the "nmn" coded above we have:

n -> letter at (position of "n" – position of "t") Mod 26 -1 = letter at position 19 = *s*
m -> letter at (position of "m" – position of "h") Mod 26 -1 = *e*
n -> letter at (position of "n" – position of "i") Mod 26 -1 = *e*

Thus, we have a *Through The Looking Glass* situation. The decoder's world is just the opposite of the coder's world. The key document plays the role of the "Looking Glass."

This code was considered to be undecipherable. However, an Englishman, Charles Babbage, known for his pioneering work in Computer Science and a Prussian Military Officer, Friedrick Kasiski independently were able to crack the code using statistical methods and frequencies of repetitions. The analysis required to decode a message in this code is much more sophisticated than that required to decode one encoded with simple substitution.

Until recently a variation of the Vigenere cipher using a different key document for every message was employed for diplomatic communications. The weak link in this code is that both the sender and receiver need to have a copy of the key document in their possession. This fact can seriously compromise the security of the code if the key document is obtained by an unauthorized agent.

The current secure method of coding, which is the basis for decoding the number at the bottom of Figure 1, involves the use of Number Theory and Abstract Algebra. We will discuss this method in detail in Procedure 3.

Now it is time to begin our exploration of coding. The code in Figure 1 turns out to be a Substitution Code. To help you crack this code, although it will not do it all for you, you will use some programs written in Derive®. These programs will help you count

frequencies of letters, and then to allow you to encode a message of your own, scramble the alphabet and encode the message using this scrambled alphabet.

FORENSIC OBJECTIVE

- To illustrate the value of evaluating evidence found at the scene of a death to help in the determination of the manner of death: natural, suicide, homicide, or accidental.

MATHEMATICS OBJECTIVES

Procedure 1 – A Simple Substitution Code

- To evaluate the frequency of occurrences of letters in a message and compare them with the known frequencies for alphabetical letters in the English Language.
- To use the distribution of the letters and knowledge of the English Language to imply the contents of a message.
- To download programs to your copy of Derive® to assist in the processes of encoding and decoding messages.

Procedure 2 – Vigenere's Code

- To compare the results of Vigenere's coding method to a simple substitution coding scheme.
- To practice using modular arithmetic.

Procedure 3 – RSA Coding

- To understand the basics of the RSA coding scheme, a public key system.
- To learn about prime numbers and their role in the RSA coding scheme.
- To note some curious properties of the MOD operation when combined with simple composite (product of two primes) numbers.

RECOMMENDED MATERIALS

- Derive® version 6.1
- A good English Language Dictionary

PROCEDURE 1 – A Simple Substitution Code

1. Following the instructions from your teacher, download the following programs to your computer.

   - Scramble
   - No_Caps
   - Encode
   - freq

   You will use these five programs to help you decode the message found on the scene (Figure 1) and to encode messages of your own. The Derive® routine MOD will be used to decode the number at the end of the message.

2. Also download the String α. This is simply an in order listing of the letters of the alphabet.

DECODING THE MESSAGE

1. Before we try the message in Figure 1, let's step through a shorter coded message using the Derive programs that we down loaded. Here are the steps from an actual Derive session.

```
#1  msg := "kdkf zbh xek vtokx nkghj, fbx kdkhyxeofl os jii gijam jfw peoxk"

#2  freq(msg)

#3 ⌈ a  b  c  d  e  f  g  h  i  j  k  l  m  n  o  p  q  r  s  t  u  v  w  x  y  z
⌉
   ⌊ 1  2  0  2  3  4  2  3  3  4  8  1  1  1  4  1  0  0  1  1  0  1  1  5  1  1
⌋
```

2. Looking at the frequency table generated by our program we see that the letter "k" is by far the most common letter (8 k's to 5 x's, the next most common letter. So we will replace "k" by "e" in the coded message,

```
"e e       e   e   e        e e                              e"
"kdkf zbh xek vtokx nkghj, fbx kdkhyxeofl os jii gijam jfw peoxk"
```

3. Nothing is too obvious at this point. You may have an idea about the third word, but returning to the frequency table where we see that "x" is the next most common letter, but not by much. Take a chance and replace "x" by "t".

```
"e e      t e    et  e       t e e  t                          te"
"kdkf zbh xek vtokx nkghj, fbx kdkhyxeofl os jii gijam jfw peoxk"
```

4. At this point we depart from the chart. We suspect that the third word is a three letter English word beginning with "t" and ending with "e", which leads us to believe that the middle letter is "h". We substitute "h" for "e".

```
"e e      the    et  e       t e e  th                       h te"
"kdkf zbh xek vtokx nkghj, fbx kdkhyxeofl os jii gijam jfw peoxk"
```

5. Returning to the frequency table for the coded message, we see "f", "j" and "o" each appear 4 times. Figure 2 notes that the next most common grouping of letters after "e" and "t" is "a", "o", "n", "r", "i", and "s". How do we decide? "j" and "f" appear together in the next to last three letter word. We suspect that they are not both vowels. This still leaves a lot of possibilities. Looking at Figure 3 we take a chance on "a" for "j" and "n" for "f". Given that replacement, "w" must be a "d" or a "y" in English. Figure 3 ranks "d" ahead of "y". So we try replacing "w" with "d".

```
"e en     the    et  e  a  n t e e  th n     a     a   and  h te"
"kdkf zbh xek vtokx nkghj, fbx kdkhyxeofl os jii gijam jfw peoxk"
```

6. We are leaving the realm where frequencies are of much use. We make one more try. "o" appears in our message 4 times. The last word would end with "hote" if we replace "o" with "o". So let's try replacing "o" with "i".

```
"e en     the   iet  e  a  n t e e  thin  i  a     a   and  hite"
"kdkf zbh xek vtokx nkghj, fbx kdkhyxeofl os jii gijam jfw peoxk"
```

7. Now it is time to rely on your knowledge of the English Language. Try and finish the quote. When you have figured out the phrase, check it with your instructor to be sure that it is correct. HINT: The substitutions that were made are correct.

8. Now it is time to try the message in Figure 1. Use the program "freq" and decode the message. When you are finished, follow the instructions in the message. What is your result?

CODE YOUR OWN MESSAGE

1. Write a message that is approximately 200 letters long to one of your friends.

2. Run the `freq` program on your message. How does the distribution of the letters compare with that in the chart of Figure 2?

3. Apply the `Encode` program to your message.

4. Run the `freq` program on the encoded program and compare the distribution of letters to the results for the original message. This will be a check on the encoding.

5. Send the encoded message to your friend. Your friend should also have a message for you.

6. Both of you can begin work on decoding each other's messages.

Procedure 2 – Vigenere Coding

1. Download the Derive® 6.1 Programs:

```
Letters_Only(param1)
Vig_Encode(param1, param2, param3)
```

   The first program requires only one parameter, the string to be processed. The second program requires 3 parameters. The first is a string for the key. The second is the message to be encoded or decoded. The third is a number to indicate whether you are encoding or decoding. 0 = decoding; 1 = encoding. NOTE: Strings are text enclosed in quotes. HINT: Assign your strings to variables and then you can simply place the name of the variable in the required position in the function call.

2. Try out the program by encoding and decoding your full name using the key: "Master Spy".

3. Use the key, "Even for the gentle zebra not everything is all black and white." to encode the message that you decoded in Procedure 1 for the note in Figure 1.

4. Apply the program `freq( )` to the newly encoded message. Compare the distribution of the frequency of letters in this message to the distribution of the frequency of letters in the original message. Are the distributions the same? Explain your answer.

Procedure 3 – RSA CODES, DIGITAL SIGNATURES, AND SECRET COMMUNICATIONS

In the high-tech world of modern business there is a need to transmit messages securely in a way that the person sending the message can easily encode it, but anyone intercepting the message other than the intended recipient can not decode it. The solution to this problem is a type of code called *Public-Key Cryptography*. This means that there is a published method of encoding that can be used by any person sending a message to a particular individual or institution can use. However, the code is a "trap door" in the sense that once it is encoded, it can not be decoded except by someone who has a

different, or private key, for decoding messages encoded with the public key. This idea was advanced in the mid 1970's and given a practical implementation a few years latter. The discoverers of the implementation were three scientists from MIT named Ronald Rivest, Adi Shamir, and Leonard Adleman. The code bears the three initials of their last names as its own name.

The trap door is this: it is very easy to multiply two numbers yet much more difficult to independently discover the factors of their product. For example, can you quickly determine the factors of 119? Yet, you can easily multiply 7 times 17. The irony of RSA coding is that it is completely breakable if you can find the factors of one of the Public Key numbers. However, the number happens to be the product of two extremely big prime numbers (Numbers containing no factors other than themselves or one, i.e. the 7 and 17 above are prime numbers). However, the primes that are chosen for this task may be 300 or more digits long. If chosen properly, it can take centuries for even the fastest computers to find the factors of their product without knowing one of the primes. Such problems, ones that can be solved, but take a very long time to solve, are called *Intractable.*

To do all of the steps you need to use the Derive® 6.1 functions, `Prime?, Next_Prime,` and `MOD`. If you are unfamiliar with these functions use the On-Line Help that is available under the Help pull-down menu.

You will also need to download another program:

```
Keys(param1, param2)
```

This program requires two parameters. Each of the parameters is a prime number.
Let's begin our encoding activity.

A HANDS-ON INTRODUCTION TO RSA CODING

1. Start by choosing two prime numbers. We will use the 7 and 17 mentioned above. Call one, p and the other q, i.e. $p = 7$ and $q = 17$.

2. Form the product of p and q. Call this product r. In our case $r = 119$. This number is part of the Public Key.

3. Next we need to find two integers, m and n, that will complete the Public and Private Keys. These integers must satisfy the relation
   $$mn = k(p - 1)(q - 1) + 1 \quad \text{for some integer, } k.$$
   The program `Keys` finds a set of values for k, m, and n. However for small primes such as $p = 7$ and $q = 17$ we will do it by hand using the Derive® functions `PRIME?` and `FACTOR`.

4. We begin testing the values for $k(p - 1)(q - 1) + 1$, for consecutive values of k, starting with $k = 1$. (1*6*16 + 1 = 97)

5. Next we test the result by applying the function `PRIME?( )`. (`PRIME?(97) = true`).

6. If the result is `false`, apply the function `FACTOR` to the result (`FACTOR(385) = 5*7*11`). From this result we can find values for *m* and *n*.

7. The following table shows the results for $k = 1,2,3$, and 4. Duplicate these results on your own.

k =	Calculation	Result
1	6*16 + 1	PRIME
2	2*6*16 + 1	PRIME
3	3*6*16 + 1	17*17
4	4*6*16 + 1	5*7*11

For $k = 1$ and 2, there are no factors for the input. For $k = 3$, we see that 17 is a factor and a possible choice for *m*. Unfortunately, if we make this choice, the value for *n* is also 17. It is not a good practice for *m* and *n* to have the same value. For $k = 4$, we have three factors 5, 7, and 11. We will choose $m = 11$ and $n = 35$. NOTE: This choice was arbitrary. We could have also chosen $m = 55$, $n = 7$, or $m = 5$, $n = 77$. Any of these choices will suit our purpose.

8. Now we are ready to code our message, but how? We represent the letters of the alphabet the same way they are in the computer. We will use the ASCII (American Standard Coded Information Interchange) code. In this system each letter of the alphabet corresponds to a number.

A	65	E	69	I	73	M	77	Q	81	U	85	Y	89
B	66	F	70	J	74	N	78	R	82	V	86	Z	90
C	67	G	71	K	75	O	79	S	83	W	87		
D	68	H	72	L	76	P	80	T	84	X	88		

There are also codes for lowercase letters, digits, and other characters.

9. To send a secret message, choose a letter, say *a*, take its ASCII code, and then transform it to a coded number, *c*, where

$$c = \text{the remainder when } a^m \text{ is divided by } r.$$

Actually, to avoid the pitfall of the substitution code, groups of letters are used instead of single letters. We ignore this point for our illustration. <u>Exercise</u>: Show that if we use only the single letters in a long message, we are no better off encoding our message than using the substitution code.

10. Suppose in our example, we wanted to send the letter, W. It's ASCII code is 87. To code it we have

$$c = \text{remainder when } 87^{11} \text{ is divided by } 119$$

We simply use the Derive function `MOD` to find c. `MOD(87^11,119)= 110.` Thus, $c = 110$. This is the value that we send.

11. To decode we do exactly the same thing to c, except that we use n instead of m. So, the original value a, in our example is:

    a = remainder when c^n is divided by 119

    Once again we used the function, `MOD`. `MOD(110^35,119)=87`.

    A result by Pierre de Fermat, the 17th Century French Lawyer who did mathematics as a hobby, guarantees that if m, n, and r are chosen as we have done, the code is completely reversible. This is the beauty of the RSA code when p and q are very large prime numbers.

12. As an exercise, take your initials; code them; and then decode them.

Using the Power of Computer Algebra and Derive® 6.1

In this section you will create two large prime numbers and use them to encode your name using the RSA Code. First we will look at the way in which alphabetical characters are coded as numbers for use in computers. The method is called the ASCII (American Standard Code for Information Interchange) Coding System. This system is designed to numerically encode every key stroke and key stroke combination that is possible using a computer's key board.

The table in Figure 4 below shows the first half of the standard ASCII 8-bit code. The code has been extended to a 16-bit code to accommodate more languages than the standard Indo-European languages.

To simplify the translation process we will apply the RSA process to messages where all of the alphabetical characters are capitalized. That means that if our message has the sentence, "This is the one.", it is transmitted as "THIS IS THE ONE." This results in the ASCII code for every character in our message being two digits.

For our next exercise, you need to down-load the following programs that are written in Derive®6.1:

```
To_Caps(msg)
Chunk_Message(msg)
RSA_Encode(msg, r, m)
Chunk_to_ASCII(num)
Decode_RSA(vec, n, r)
```

As you can see, each of the programs require some parameters. You can guess that r, m, and n are associated with the Public and Private Keys that you will create to encode and decode your message. The parameter *msg* is the message you will be encoding. The parameter *vec* is a vector resulting from the encoding of the numbers associated with the numerical form of your message.

Dec	Hex	Sym	Dec	Hex	Char	Dec	Hex	Char	Dec	Hex	Char
0	0	NUL	32	20		64	40	@	96	60	`
1	1	SOH	33	21	!	65	41	A	97	61	a
2	2	STX	34	22	"	66	42	B	98	62	b
3	3	ETX	35	23	#	67	43	C	99	63	c
4	4	EOT	36	24	$	68	44	D	100	64	d
5	5	ENQ	37	25	%	69	45	E	101	65	e
6	6	ACK	38	26	&	70	46	F	102	66	f
7	7	BEL	39	27	'	71	47	G	103	67	g
8	8	BS	40	28	(	72	48	H	104	68	h
9	9	TAB	41	29	)	73	49	I	105	69	i
10	A	LF	42	2A	*	74	4A	J	106	6A	j
11	B	VT	43	2B	+	75	4B	K	107	6B	k
12	C	FF	44	2C	,	76	4C	L	108	6C	l
13	D	CR	45	2D	-	77	4D	M	109	6D	m
14	E	SO	46	2E	.	78	4E	N	110	6E	n
15	F	SI	47	2F	/	79	4F	O	111	6F	o
16	10	DLE	48	30	0	80	50	P	112	70	p
17	11	DC1	49	31	1	81	51	Q	113	71	q
18	12	DC2	50	32	2	82	52	R	114	72	r
19	13	DC3	51	33	3	83	53	S	115	73	s
20	14	DC4	52	34	4	84	54	T	116	74	t
21	15	NAK	53	35	5	85	55	U	117	75	u
22	16	SYN	54	36	6	86	56	V	118	76	v
23	17	ETB	55	37	7	87	57	W	119	77	w
24	18	CAN	56	38	8	88	58	X	120	78	x
25	19	EM	57	39	9	89	59	Y	121	79	y
26	1A	SUB	58	3A	:	90	5A	Z	122	7A	z
27	1B	ESC	59	3B	;	91	5B	[	123	7B	{
28	1C	FS	60	3C	<	92	5C	\	124	7C	|
29	1D	GS	61	3D	=	93	5D	]	125	7D	}
30	1E	RS	62	3E	>	94	5E	^	126	7E	~
31	1F	US	63	3F	?	95	5F	_	127	7F	

Figure 4: The first half of the 8-bit ASCII code

A demo using large primes

1. We will find two large primes that are at least 14 digits in length. To do this we choose four dates having some historical importance

14 July 1789	Bastille Day	14071789
4 July 1776	U.S. Declaration of Independence	4071776
15 June 1215	Magna Carta	15061215
12 Sept 1683	Battle of Vienna	12091683

2. We create the primes from these four numbers using the Derive function, `Next_Prime()`. We compute `Next_Prime(14071789*4071776)` and `Next_Prime(15061215*12091683)`. These will be the primes, p and q that are used to construct r , one of the numbers in the public key.

3. Using *p* and *q* as the input parameters for the program, `Keys()`, run `Keys(p,q)` to obtain *r, m, n*.

4. We will encode the message "Mathematics Rules". Recall that the public code will be r and m. You can go directly to the encoding by running `RSA_Encode ("Mathematics Rules",r,m)`. However, it is instructive to see what is being encoded. Run `Chunk("Mathematics Rules")`. A choice was made to break the message into "chunks" of seven letters plus what was left as a remainder. Check each of the chunks with the ASCII coding table.

5. Now run `RSA_Encode("Mathematics Rules",r,m)`. The result is *vec*, the vector of numbers that you will send as your message.

6. Check that your message was correctly encoded. Run `Decode_RSA(vec, n, r)` and read the message.

7. In order to show you that there is a need to use extremely large primes for this code to be secure. Enter `FACTOR(r)` in Derive and simplify it. How long does it take Derive to factor this 28 digit number? A person with Derive can easily crack the "Public Key" given in this exercise and decode your message. The process is not secret, but the use of extremely large primes (300 digits or more) makes it "intractable".

8. Choose four dates that are important to you, e.g. your birth date, your parents' birth dates, etc. and create a public and private key using the Derive programs that you have down-loaded. Have a friend encode a short message for you using your public key and then decode it.

CORONER'S COMMENTS

While this activity deals with a code and secret bank account, the fundamental forensic issue is that it is necessary to carefully investigate the scene of a death even when the result of the investigation seems obvious. Even innocuous cases can end up in court and your findings challenged. In this case an autopsy revealed that the number of pills found in the stomach of the victim far exceeded any possibility that the overdose was due to an unintentional mistake. The death was ruled suicide and the result was supported by legal opinion.

Unfortunately, some families do alter the scene of the death in order to thwart the investigation and obtain a ruling that will increase their insurance payment. This is outright insurance fraud and must be dealt with in the severest possible manner.

SPEED KILLS

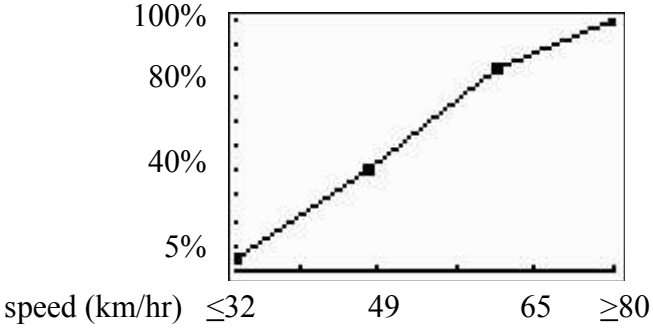
INVESTIGATING PEDESTRIAN STRUCK ACCIDENTS

SCENARIO

The call had come in to the Coroner's Office for a pedestrian struck. You are assigned to investigate the accident. When you arrive on the scene, the cause of death is obvious to you. The vehicle that had struck the victim was 15 feet beyond where the victim came to rest. The vehicle, a late model full size pick-up had damage to the right front headlight area. When you examined the victim, you note that he is a 35 year old male, who is fully dressed and his shoes are still on. He is lying face down on the ground and had slid from the initial point of impact along the ground to his final resting location. There are obvious fractures to both legs and contusions on his lateral left upper leg and at the level of his lower ribs. The posted speed limit was 35 miles per hour. It was the duty of the police at the scene to determine if the driver of the truck was speeding.

BACKGROUND

In the year 2003 the U.S. National Highway Traffic Safety Board reported that over 70, 000 pedestrians were injured as a result of being struck by a motor vehicle and 4,749 were killed. Of all of the children aged 5 – 9 killed in traffic accidents 22% were a result of pedestrian struck accidents. For youths under the age of 16 killed in traffic accidents 17% were killed as a result of pedestrian struck accidents. The following figure relates the percentage of individuals involved in pedestrian struck accidents that were killed and the speed of the vehicle that struck them.



Obviously, speed is an important factor in determining the outcome of a pedestrian struck accident. That is why it is important to determine the speed of the vehicle that struck the person. However, The question of vehicle speed based on only the limited data available at the accident scene is one of the hardest of all questions for an accident reconstructionist to answer.

The principle is simple. The momentum, p, of a moving object is given by the formula:

$$p = mv$$

where m = mass of the object and v = velocity of the object. Furthermore, even in an accident involving a tractor-trailer truck and a pedestrian, momentum is conserved. The problem is to figure out how much momentum was lost by the vehicle. For example, a crucial factor for determining the transfer of momentum is how the pedestrian was struck. Was the point of contact above, at, or below the pedestrian's center of gravity? Another factor is where on the vehicle was the person struck? Was it in the center of the vehicle or at a fender? Other factors are: the size and weight of the vehicle? the size and weight of the pedestrian? All of these factors enter into the dynamics for reconstructing the accident.

One tool used by accident investigators is the distance that the pedestrian has been thrown. There is, however, no consensus on how to include this factor. This has been an active area of research since 1975. Several formulas have been proposed. The earliest related distance thrown to the square of the velocity, another had it related to a linear term plus a cubic term, other formulas include using the launch angle (a very difficult parameter to obtain). One consensus is that children tend to be thrown further than adults. Another is that pedestrians struck by high fronted vehicles tend to be thrown further than those struck by low fronted vehicles.

In this activity you will investigate the first of the claims made at the end of the previous paragraph; that children tend to be thrown further than adults. You will release a toy vehicle or a dynamics cart from different heights on a ramp to strike an object at the bottom of the ramp. There will be two trials at each height, each with a different object at the bottom of the ramp. One of the objects represents an adult and the other will be a lighter version of a similar object representing a child. You will observe the distance and position of the object after the impact. The impact will occur with the right fender area of the vehicle. After you have gathered your data, you will do a power regression to see how well your data relates the distance thrown to a power of the velocity.

FORENSIC OBJECTIVES

- To investigate the effect of a vehicle striking an individual whose mass is much less than that of the vehicle.

- To investigate the difference in the distance that a lighter object is thrown to the distance a heavier object is thrown by the same vehicle traveling at the same speed.

SCIENCE/MATHEMATICS OBJECTIVES

- To accurately measure the distance and angle an object is thrown when struck by a moving vehicle.

- To systematically record and analyze the data gathered from several different "pedestrian struck" accidents.

- To test several hypotheses concerning the relationship between distance thrown and the speed of the vehicle.

MATERIALS

- Derive® 6.1, a Voyage200, or a TI-89
- Vernier Dynamics Kit with ramp and cart (option 1)
- A piece of corrugated board 90 cm X 30 cm (option 2)
- A toy truck approx 8.5 cm tall weighing approx 4.5 N (option 2)
- A piece of wooden doweling 2.54 cm X 11.5 cm
- A piece of wooden doweling 1.6 cm X 8.5 cm
- 2 pieces of styrofoam swimming noodle cut to a length of 5 cm
    - one piece trimmed to a square cylinder 4cm X 4cm with hole in center for the larger dowel
    - one piece trimmed to a square cylinder 3cm X 3cm with hole in center for the smaller dowel
- A meter stick or a tape measure showing metric units
- Enough books to make a ramp about 30 - 35 cm high (approx 5 to 6 books)
- Data Record Sheet

PROCEDURE

1. Place the large dowel in the 4 X 4 styrofoam block with the bottom of the dowel at the bottom of the center hole. Place the smaller dowel in the 3 X 3 styrofoam block so that the bottom of the dowel is a little more than half way down the hole in the center of the block. Both figures should be about the same overall height.

2. If you are using a corrugated board ramp, draw and label lines across the ramp at the following distances from the end of the ramp: 25 cm, 35 cm, 50 cm, 70 cm, 80 cm, and 85 cm. Also draw lines the length of the ramp and perpendicular to the lines you have just drawn. These lines will denote the path of the vehicle's wheels down the ramp.

3. Place the books to elevate the ramp and carefully place the ramp (either corrugated board or Dynamics Kit track) with the 85 cm mark at the edge of the books. Be sure the ramp is in this position prior to each trial. If you are using the Dynamics Kit have the bumper in the center of the track installed at the bottom of the incline.

4. Measure the distance of the bottom of the incline from the books that support the top of the ramp.

5. Since you know this horizontal distance and the length of the incline (85 cm), you can compute the angle, $\angle\alpha$, at the base of the incline using the $\cos^{-1}$ function on your calculator.

6. Using g as the acceleration due to the force of gravity (9.8 m/sec^2), the acceleration of the vehicle along the incline is: $g \cdot \sin(\alpha)$. This yields the following equations for the motion of the vehicle down the ramp

$$a = g \cdot \sin(\alpha)$$

$$v = t \cdot g \cdot \sin(\alpha)$$

$$d = \tfrac{1}{2} t^2 \cdot g \cdot \sin(\alpha)$$

where

a = acceleration along the incline
t = time traveled along the incline
v = velocity of the vehicle at time, t
d = distance traveled by the vehicle at time, t

7. The above equations can be used to find the velocity of the vehicle at the bottom of the incline after it has traveled a distance, d. Use these formulas to calculate the velocity of the vehicle after traveling distances of 80, 70, 50, 35, and 25 cm.

8. Record these velocities on your data sheet.

9. Place the figure so that it straddles the path of the right wheel. Mark the placement so that it can be used for each trial. This will simulate a strike by the right bumper below the center of gravity.

10. Position the front of the vehicle at the 80 cm mark and release the vehicle.

11. Measure and record the distance of the bottom of the figure from its original (standing) position in cm.

12. Repeat steps 9 – 11 for the smaller (child) figure.

13. Repeat steps 9 – 12 for distances of 70 cm, 50 cm, 35 cm, and 25 cm.

Analysis of the Data

Unlike many of the previous experiments, we will not be using a predetermined mathematical equation or formula to analyze our data and come to a conclusion about the evidence found at the scene of an accident or crime. Instead, we will be "experimenting" with different mathematical models based on the data. We will include a partial analysis of two experiments that were done using the materials described in the procedure section of this discussion.

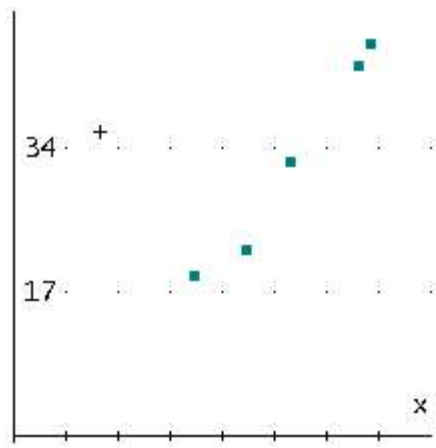
In each case, after you duplicate the "experiments" using your CAS, you will be asked which of the models make the most sense for describing your experimental results. What hypotheses you are making by choosing a particular model? Compare your data with that in the examples and also with that collected by your classmates. Do these comparisons affect your opinion on what model best describes the phenomenon of a pedestrian struck by a moving vehicle? Of what use are these measurements?

Derive® 6.1 Session Using the Data For the Adult Pedestrian Struck Simulation

Data from an experiment with a (plastic about 12cm tall) toy truck and an "adult" sized doweling (2.54 cm diameter X 11.5 cm length set in the center of a piece of styrofoam 4 cm X 4 cm X 5 cm). The bumper of the truck should meet the "leg" area of the doweling.

$$\#1: \quad Dat := \begin{bmatrix} 0.869 & 19 \\ 1.119 & 21.9 \\ 1.33 & 32.4 \\ 1.65 & 43.7 \\ 1.708 & 46.3 \end{bmatrix}$$

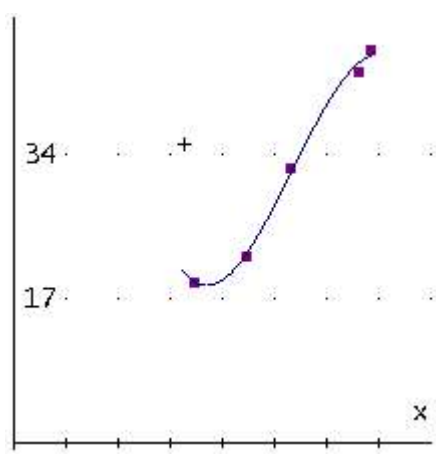
We plot the data to get a "feel" for its shape.



It looks like a parabola or a cubic curve may be a good fit. Let's try a cubic curve fit.

#2: $\text{APPROX}(\text{FIT}([x, a\cdot x^3 + b\cdot x^2 + c\cdot x + d], \text{Dat})) = (-95.63074969\cdot x^3 + 384.0331383\cdot x^2 - 464.6853727\cdot x + 195.4615397)\cdot \text{IF}(x > 0.8 \wedge x < 1.75, 1)$

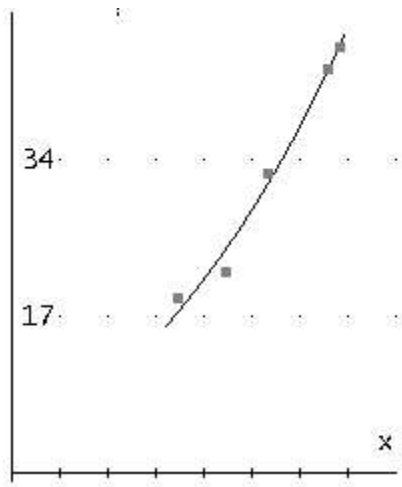
We only want to describe the reaction, distance thrown, to the domain of our data. Thus, we restrict the graph to this domain using the IF statement that is available in Derive.



One of the proposed models for a fit was to use a cubic term plus a linear term for relating the distance thrown. Assuming that the curve is increasing, it tends to be concave upward in the first quadrant. NOTE: The type of curve we are discussing passes through the origin and is either always increasing or has one maximum and one minimum on either side of the origin. That property will most likely eliminate the local minimum and maximum that we see in the above graph. The result should be a graph more in line with our expectations.

#3: $\text{APPROX}(\text{FIT}([x,\ a\cdot x^3 + c\cdot x],\ \text{Dat})) = 3.215028742\cdot x^3 + 17.75377204\cdot x$

#4: $(3.215028742\cdot x^3 + 17.75377204\cdot x)\cdot \text{IF}(x > 0.8 \wedge x < 1.75,\ 1)$



This graph appears to match our intuition about how the result of a lighter body being impacted by a larger vehicle would react. But is it the best answer?

Perhaps, an exponential curve would fit our data. To do an exponential fit in Derive, we do a straight line fit to the logarithm of the data:

if $\quad LN(y) = a + b*x$

then

$$y = EXP(a + b*x) = C*EXP(b*x) \qquad (C = e^a)$$

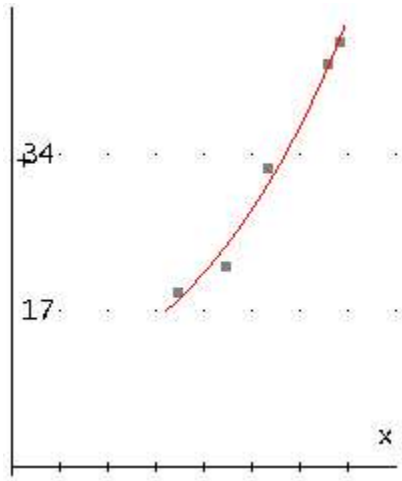
#5: $\text{LDat} := \begin{bmatrix} 0.869 & LN(19) \\ 1.119 & LN(21.9) \\ 1.33 & LN(32.4) \\ 1.65 & LN(43.7) \\ 1.708 & LN(46.3) \end{bmatrix}$

#6: $\text{APPROX}(\text{FIT}([x,\ a + b\cdot x],\ \text{LDat})) = 1.116642954\cdot x + 1.933373146$

#7: $\text{APPROX}(\text{EXP}(1.116642954\cdot x + 1.933373146)) = 6.912788804\cdot e^{1.116642954\cdot x}$

#8: $6.912788804\cdot e^{1.116642954\cdot x}\cdot \text{IF}(x > 0.8 \wedge x < 1.75,\ 1)$

We restrict the curve to the domain of the data.

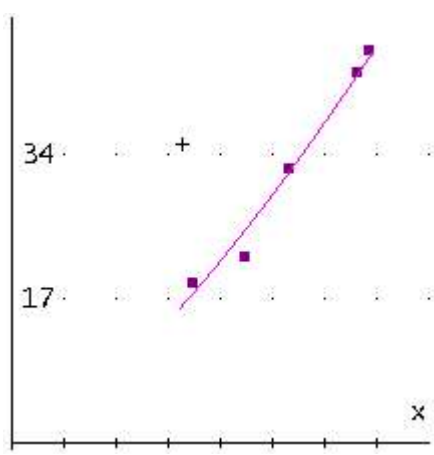


Next, we try a power function curve fit to our data. NOTE:
If LN(y) = a + b*LN(x)
then $y = A * x^b$
We are fortunate that Derive can do least squares curve fitting with the LN function (as well as SIN, COS, ATAN, EXP, etc.) in addition to powers of x as long as the relationship is a linear one.

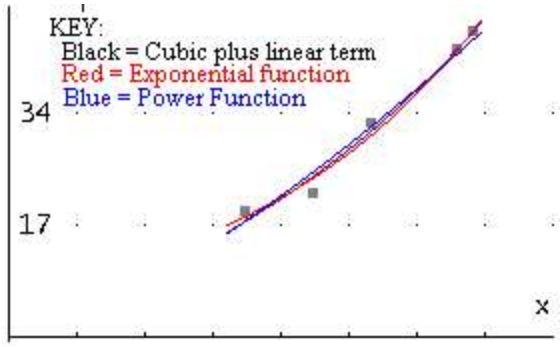
```
#9:   APPROX(FIT([x, a + b·LN(x)], LDat)) =
                    1.394815775·LN(x) + 3.063531604

#10:  APPROX(EXP(1.394815775·LN(x) + 3.063531604)) =
                                        1.394815775
                            21.4030108·x
```

Again we restrict the domain of the independent variable, speed of the vehicle, and graph.

```
                    1.394815775
#11:  (21.4030108·x            )·IF(x > 0.8 ∧ x < 1.75, 1)
```



Below is a graph with results of the last three of the curve fitting procedures.



1. Fit a parabolic curve of the form $ax^2 + bx$ to the original data. How does this curve fit compare with the other fits we have done?
2. Which of the fits do you think best describes the distance the person is thrown vs. the speed of the vehicle? Why?
2. It will be a natural question for someone looking at your choice of a model to ask you to explain the meaning of the constants in your model. Give an explanation in terms of the phenomenum that your model (your answer for #1) is describing.
3. Is it always the best policy to use the curve that has the closest least squares fit to the data?
4. Each of the models only have one independent variable, the distance the person is thrown. If you were asked to testify in a court case involving such an accident, would you feel comfortable in using your choice of a model to give an approximation of the speed of the vehicle at the time of the impact? Why?
5. What other factors should you consider when you are constructing a mathematical model for your observations? How might you incorporate them into your model?

Derive® 6.1 Session Using the Data For the Child Pedestrian Struck Simulation

The data collected using the same vehicle as in the case of the adult simulation. However, this time the pedestrian struck was a "child" doweling (1.6 cm wide X 8.5 cm in length placed in a piece of styrofoam 3cm X 3cm X 5 cm). The bumper of the vehicle hits the child in the "upper thigh" or "waist" region.

$$\#1: \quad Dat := \begin{bmatrix} 0.896 & 23 \\ 1.119 & 40.7 \\ 1.33 & 70.2 \\ 1.65 & 73.8 \end{bmatrix}$$

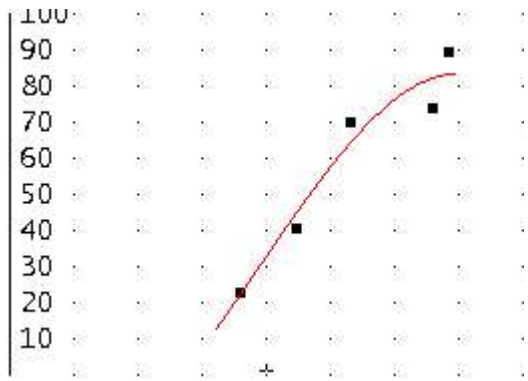
$$\lfloor\ 1.7082 \quad 89.2\ \rfloor$$

We plot this data and not that it is not as nicely distributed as the adult data. At issue are the third and fourth observations. For these two speeds in our experiment, the child was thrown about the same distance. We follow our procedure of trying three curve fits. The first is a cubic curve.

$$\#2: \quad \mathrm{APPROX}(\mathrm{FIT}(\left[x,\ a\cdot x^3 + b\cdot x^2 + c\cdot x + d\right], \mathrm{Dat}))$$

$$= -\ 58.93888316\cdot x^3 + 172.0043277\cdot x^2 - 62.99001123\cdot x - 17.3181781$$

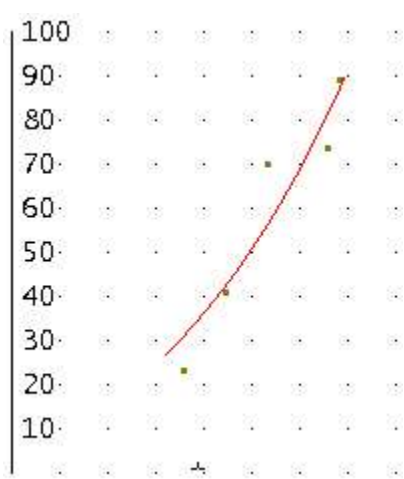
$$\#3: \quad (-\ 58.93888316\cdot x^3 + 172.0043277\cdot x^2 - 62.99001123\cdot x - 17.31817813)\cdot \mathrm{IF}(x > 0.8 \wedge x < 1.75,\ 1)$$



This graph appears to be heading for a local maximum at the end of our data set. Let's look at the cubic curve consisting of only a cubic term and a linear term.

$$\#4: \quad \mathrm{APPROX}(\mathrm{FIT}(\left[x,\ a\cdot x^3 + b\cdot x\right], \mathrm{Dat})) = 7.691272051\cdot x^3 + 28.53684118\cdot x$$

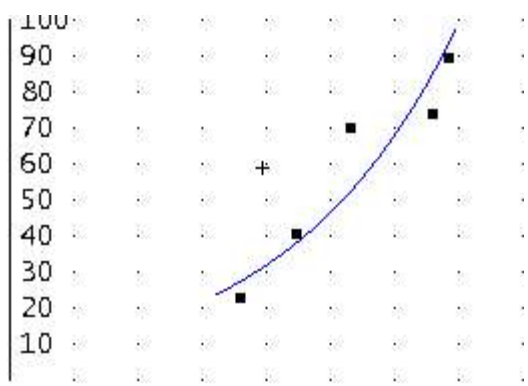
$$\#5: \quad (7.691272051\cdot x^3 + 28.53684118\cdot x)\cdot \mathrm{IF}(x > 0.8 \wedge x < 1.75,\ 1)$$



Next is an exponential curve. Recall, we need to take the logarithm of the dependent variable.

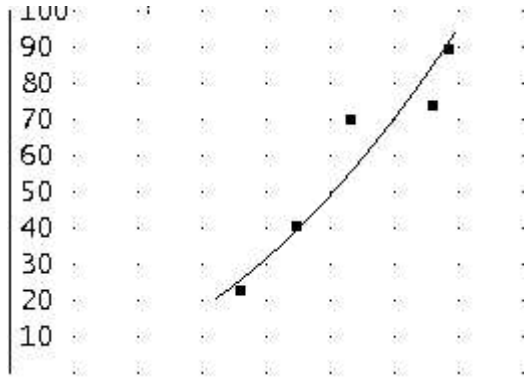
$$\#6: \quad LDat := \begin{bmatrix} 0.896 & LN(23) \\ 1.119 & LN(40.7) \\ 1.33 & LN(70.2) \\ 1.65 & LN(73.8) \\ 1.7082 & LN(89.2) \end{bmatrix}$$

```
#7:       APPROX(FIT([x, a + b·x], LDat))
                     = 1.508097177·x + 1.955246677

#8:   APPROX(EXP(1.508097177·x + 1.955246677))
                                   1.508097177·x
                     = 7.065661744·e

                    1.508097177·x
#9:   (7.065661744·e             )·IF(x > 0.8 ∧ x < 1.75, 1)
```

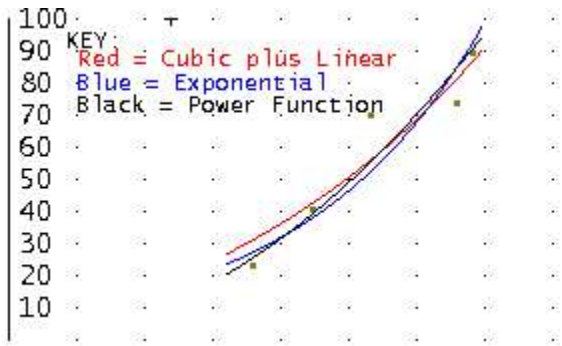


Again, we look at a power function curve fit.

```
#10:  APPROX(FIT([x, a + b·LN(x)], LDat))
                = 1.972382917·LN(x) + 3.454769172


#11:  APPROX(EXP(1.972382917·LN(x) + 3.454769172))
                                  1.972382917
                     = 31.6509819·x

                   1.972382917
#12:  (31.6509819·x           )·IF(x > 0.8 ∧ x < 1.75, 1)
```



Finally we look at the data and the three curve fits on the same graph



1. Which curve do you think best describes the data?
2. Is this the same choice that you made in the case of the adult pedestrian struck? Explain your answer.
3. Why are the functions for the child sized dowling and adult sized different? Do you think this would be the case in the real world situation?

Suppose that we eliminate the data point for the velocity of 1.33 cm/sec and repeat our analysis.

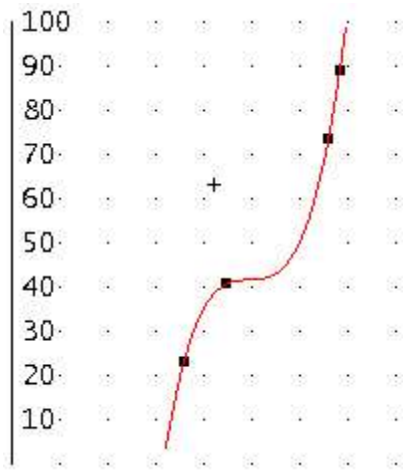
$$\#13: \quad Dat := \begin{bmatrix} 0.896 & 23 \\ 1.119 & 40.7 \\ 1.65 & 73.8 \\ 1.7082 & 89.2 \end{bmatrix}$$

$$\#14: \quad APPROX(FIT([x,\ a\cdot x^3 + b\cdot x^2 + c\cdot x + d],\ Dat)) = 450.4934392\cdot x^3 - 1673.65392\cdot x^2 + 2074.355655\cdot x - 816.0368756$$

$$\#15: \quad (450.4934392\cdot x^3 - 1673.65392\cdot x^2 + 2074.355655\cdot x - 816.0368756)$$

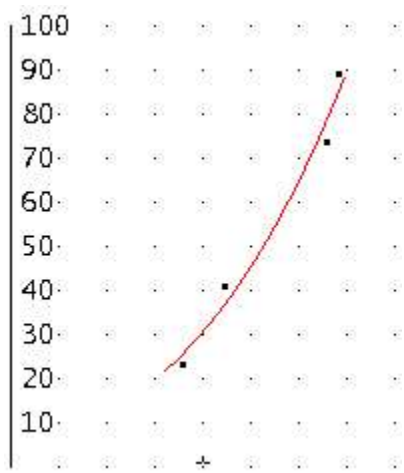
·IF(x >

0.8 ∧ x < 1.75, 1)



#16: $\text{APPROX}(\text{FIT}([x,\ a\cdot x^3 + b\cdot x],\ \text{Dat})) = 10.02213102\cdot x^3 + 20.59702927\cdot x$

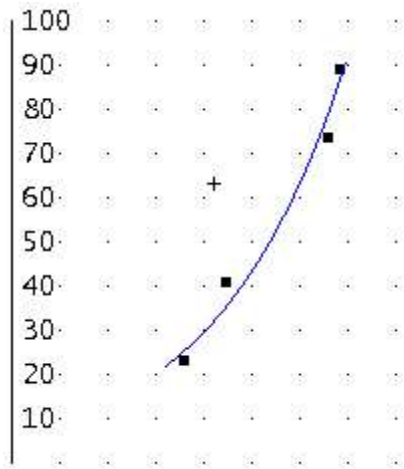
#17: $(10.02213102\cdot x^3 + 20.59702927\cdot x)\cdot \text{IF}(x > 0.8 \wedge x < 1.75,\ 1)$



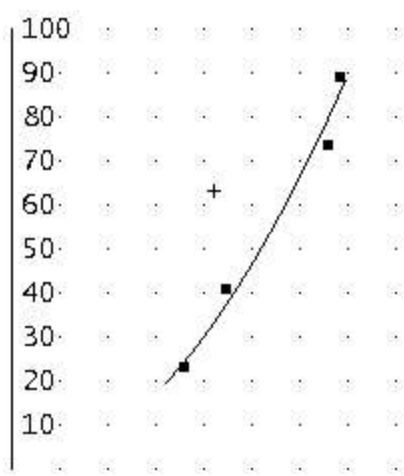
#18: $\text{LDat} := \begin{bmatrix} 0.896 & \text{LN}(23) \\ 1.119 & \text{LN}(40.7) \\ 1.65 & \text{LN}(73.8) \\ 1.7082 & \text{LN}(89.2) \end{bmatrix}$

#19: $\text{APPROX}(\text{FIT}([x,\ a + b\cdot x],\ \text{LDat})) = 1.516182149\cdot x + 1.871803038$

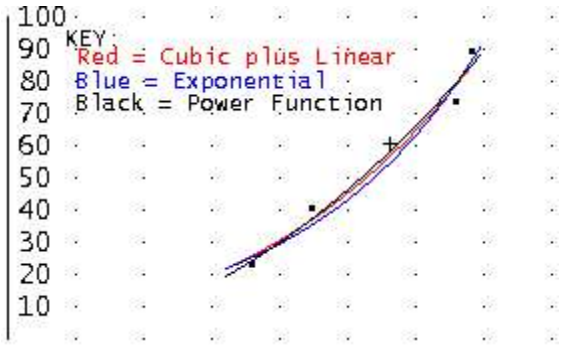
```
                                                                    1.516182
148·x
#20:  APPROX(EXP(1.516182149·x + 1.871803038)) = 6.500005597·e

                    1.516182148·x
#21:  (6.500005597·e             )·IF(x > 0.8 ∧ x < 1.75)
```




```
#22:  APPROX(FIT([x, a + b·LN(x)], LDat)) = 1.951990014·LN(x) +
3.401541127


#23:  APPROX(EXP(1.951990014·LN(x) + 3.401541127)) =

                     1.951990014
        30.01031413·x

                    1.951990014
#24:  (30.01031413·x           )·IF(x > 0.8 ∧ x < 1.75)
```



Looking at all three on the same graph, we can see the differences in the three results.



4. Compare all five of the graphs that you generated using the complete data set and after eliminating one of the data points. Note the differences that you observe in the graphs.
5. What is your opinion about removing a data point from the analysis? Do you think that it is ethical to do this? Under what circumstances would you consider such a deletion permissible?

Now it is time for you to enter your data and do your analysis. If you have another candidate for a model, please try it out.

CORONER'S COMMENTS

There are a number of factors that must be taken into consideration when determining the speed of the striking vehicle. Depending on the speed, size of victim, size and weight of the vehicle, and the point of the vehicle that struck the pedestrian, the victim may go forward, fly to the side, land on the hood and roll off at some point, or impact with the windshield and continue over the vehicle landing well behind the impacting vehicle.

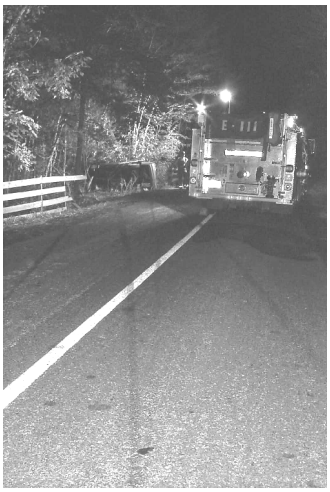
It is necessary for the investigator to completely document the answers to the following questions:

1. Did the victim hit the windshield?
2. Is there a dent in the roof?
3. Was there any impact with the trunk of the vehicle?
4. Is there evidence of braking? If so, look for fender drop. Fender drop is extremely important if this is a hit and run accident. Fender drop can be analyzed as to where the vehicle has been prior to the incident.
5. Was the victim knocked out of his or her shoes?
6. Was the victim still in his or her clothing or were they shredded as a result of dragging along the roadway?
7. Are there impact marks on the vehicle?
8. Is there hair, blood or tissue on the vehicle?
9. Where was the impact on the victim? (i.e. right leg, left leg, pelvis, front or back of victim) This can help to determine whether the victim was walking, where the victim may have been walking, or if the victim standing or lying on the roadway.
10. Is there any evidence of the vehicle on the victim such as paint chips or glass?

11. Were there obstructions to the driver's view such as bright sun light, the crest of hill, or parked cars?
12. What is the posted speed limit?

Unless you have reliable witnesses to the accident, all of the above factors have to be considered in order to determine the how, what, why and where of this accident and who was at fault.

Investigating Motor Vehicle Accidents



SCENARIO

As an investigator for the Coroner's Office you were called to the scene of a single vehicle motor crash. When you arrived at the scene you observed a single vehicle on its side against a tree. You noted a very long skid mark, but because it is dark and your available lighting was limited, you were only able to photograph portions of the skid. The victim in this vehicle had been ejected because he had not been wearing a seat belt. Upon further investigation and interviews the following facts came to light. The 19 year old male victim had come to a friend's party with his girlfriend. Alcohol was being served even though no one was of legal age. While at the party the victim had an argument with his girlfriend, and he jumped into his car and left traveling at a high speed. The victim had only driven about a mile, came around a mild curve and the vehicle went out of control and off the side of the road. He was not an experienced driver and had been drinking and overcorrected the steering which caused his car to cross the center line where his car began to skid. The vehicle turned sideways and when it came to the opposite edge of the road began to roll. The vehicle rolled over sideways at least three times and during one of the rolls the victim was ejected. The vehicle came to rest on the passenger side wheels against a tree. As can be seen in the photo, the vehicle was a listed as a total loss.

BACKGROUND INFORMATION

Automobile tires play a very important role in the operation and safety of our motor vehicles. The frictional force generated by the tire against the surface of the road affects the way our cars handle and also controls the amount of time that it takes us to bring the car to a safe stop. However, this force produces wear on the tire and gradually eats away

the tire tread. So friction, or drag, has good effects, but at a cost. In this activity we will examine how the extreme wearing of the tire, or skid marks on the road, that results from a driver attempting to bring a vehicle to an abrupt stop can give investigators evidence of the speed at which the vehicle was traveling when the brakes were applied.

When a driving emergency arises, it is a natural reaction of an inexperienced driver to press hard on the brake pedal. This is, generally, a bad thing to do. It will propel the car into a skid which is extremely, if not impossible, to control. An experienced, and calm, driver knows to apply the brakes in a more gentle and staccato manner to bring the car under control. Unfortunately, if the emergency is extreme, or the driver is impaired from alcohol or drug consumption, or the driver is just plain inexperienced, it is almost impossible to remain calm and avoid a skid. In most cases the end result is poor and, far too often, results in one or more deaths.
d = distance traveled

In order to find the relationship of the vehicle speed to the length of a skid mark, we need to go to Newton's Laws of Motion and also the equation for frictional force. Newton's basic Laws give us the following formula for the Kinetic Energy of the Vehicle.

$$KE = \frac{1}{2}mv^2$$

where

$$KE = kinetic\ energy$$

$$m = mass\ of\ the\ vehicle = \frac{weight\ of\ the\ vehicle}{acceleration\ of\ gravity} \approx \frac{w}{9.88}$$

$$v = velocity\ of\ the\ vehicle$$

However, the Kinetic Energy is also related to the frictional force by the equation

$$KE = w \cdot f \cdot d$$

where

$$w = weight\ of\ the\ vehicle$$

$$f = coefficient\ of\ friction$$

$$d = distance\ traveled$$

If we equate these two expressions of Kinetic Energy for a skidding vehicle and do some algebraic manipulations to solve for *v*, we can relate the distance the car skidded to its pre skid velocity.

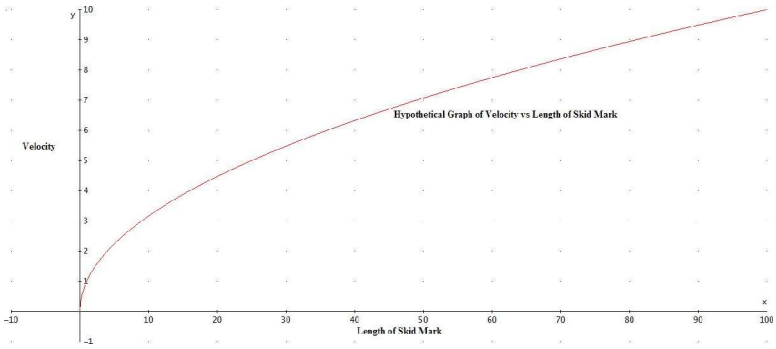
You will be asked to derive this equation on your work sheet for this activity.
, the velocity of the vehicle at the start of the skid, we have the following expression equating the velocity to the length of the skid marks.

$$v = \sqrt{2g \cdot f \cdot d} \qquad (1)$$

Equation (1) is a rather surprising result! The weight of the vehicle has no effect on determining the velocity of the vehicle from the length of the skid marks. The only

effects are the acceleration of gravity and the coefficient of friction which is related to the tires on the vehicle.

Thus, given a value for the coefficient of friction, we can draw a graph of the relationship of the velocity of the vehicle at the start of a skid and the length of the skid marks. This velocity has a shape similar to the following graph.



It would be very difficult (and very dangerous for both the driver as well as the data collector) to produce skid marks for an actual automobile skid. In place of that we will simulate a skid using a toy automobile and an inclined plane. After rolling down the plane, the only force acting upon the car, which has no motor, will be a frictional force. Using the equation relating velocity and the distance traveled, we can calculate the coefficient of friction for this force.

The main experimental tool that we will be using is the CBR2® Calculator-Based Ranger® and our TI hand-held device.

After using the information obtained about the coefficient of friction for your "experimental vehicle," you and the members of your team will produce a graph of the distance traveled by the vehicle and the speed the vehicle was traveling when it was released to roll on its own. Your team will conduct several trials and determine the speed at which the vehicle was released after being pushed by a team member.

FORENSIC OBJECTIVES

- To determine the speed at which a vehicle was traveling prior to applying the breaks and locking the wheels of the vehicle at the scene of a motor vehicle accident.

- To determine the minimum stopping distance for a vehicle traveling at a particular speed.

MATHEMATICAL AND SCIENTIFIC OBJECTIVES

- To determine the coefficient of rolling friction for a particular vehicle.

- To find the relationship between velocity and stopping distance.

- To create a graph showing this relationship

MATERIALS

Toy Car – minimum height 1 - 1½", without friction motor and preferably no technology to reduce the friction of the axels in their guides
2 or 3 thin books to make a platform about 3" high
A thin piece of cardboard or wood about 10 – 12" long (see pictures at the end of the Background Section)
A Tape Measure (optional – the CBR can be used for most measurements)

PROCEDURE 1 Using a CBR – Requires Algebra Only

Determining the Coefficient of Friction

1. On the floor of your classroom or on a lab table set up the experiment as indicated in the photo at the end of the Background Information section.

2. Plug in the EasyLink® cable. The Main Screen of the EasyData® Application should appear after the EasyData® Welcoming Screen. At the top is a box that shows the distance of the closed object from the sensor.

3. If the Distance is not being measured in feet (the alternative is meters). Press the grey button labeled Window at the top of the calculator and under the "Setup" tab displayed on the EasyData® screen of the calculator. This will display a pop up menu. Choose the "1. Distance" option by pressing the Enter button. Press the grey Zoom button under the "Units" tab and choose option "2. (ft)". To exit to the main screen choose the "OK" tab.

4. Perform a "Test Run" with your car. Start with the back wheels at the top edge of the incline and release it. Have another team member check the readings on the TI handheld device. You do not have to have EasyData® recording any of the motion at this point. Be sure that the CBR® is able to pick up the distance of the car from the CBR® starting at the time of its release until it stops. Also, be sure that the CBR® is positioned so that the car stops a minimum of 6" in front of it.

5. Now you are set to begin the experiment. Under the "Setup" tab choose the option "2. Time Graph" Under the "Edit" tab make the following settings:
   Sample Interval: .05 seconds – this is the minimum sampling time
   Number of Samples: 40
   Experiment Length (s): 2
   Actually, two seconds is longer than the experiment will take (assuming you do not have a really high incline); however, it helps to distinguish the motion of the car if you hold it at the top of the incline for about a half of a second.

6. When all members of your team are ready and you have checked the experimental setup, press the button under the "Start" tab and release the car from the top of the incline towards the CBR2®. It must have stopped rolling before the screen reads "Transferring Data".

7. Look at the graph that appears on the screen. Is it fairly smooth? If it has large jumps, that means you have picked up a lot of "noise" in your data. Recheck your experimental set up and rerun your test again.

8. If your distance graph is acceptable to you, press the button under the "Plots" tab and choose option "2. Vel vs Time" and make sure the velocity graph is smooth

and readable. If not, it is best to recheck your equipment and rerun the experiment.

9. Now that you have a good distance graph and velocity graph, start at the velocity graph. Since the motion of the car is towards the CBR2® the majority of the readings should be negative. Trace the graph to the point where the reading is the negative number with the largest magnitude (the lowest point on the graph). This is the velocity of the car at the bottom of the incline. It starts slowing down as a result of the frictional force after that.

10. Press the button below the "Plots" tab and choose option "1. Distance". The crosshair will still be at the same position on the time axis. That is the distance of the car from the CBR® when it is at the bottom of the incline.

11. Return to the Velocity Graph and trace to the point where the velocity is as close to zero as possible (because of the time between samples, the graph may jump from negative to positive in between two of the data points. Move to the Distance Graph and note the distance. This is where the car stopped.

12. Subtracting the distance obtained in step 11 from that in step 10, you have the distance that the car traveled under the influence of the frictional force alone.

13. Using the absolute value of the velocity of the car at the base of the incline (step 9), find the value for the coefficient of friction using formula (1) given in the **Background Information**. Write this value down since you will be using it in the next part of the experiment.

14. Using this value for the coefficient of friction and your TI-83 Plus or TI-84 Plus draw a graph of starting Velocity vs Distance for the car that is moving only under the influence of the frictional force. Allow for distances up to six feet in length.

Using Your Findings to Determine Starting Velocities

1. Using a piece of tape at least 18" long, or a straight edge and a graphite pencil, draw a "starting line" on your lab table or on the floor where you were working for the first part of this activity.

2. Place the CBR® so that the "beam" from the sensor is perpendicular to this line. The CBR® should be at least four or five feet away from the starting line.

3. Place a book or a file folder so that the face closest to the CBR® is on the starting line and in line with the sensor.

4. Note the distance of the book or file folder from the sensor face on the CBR®.

5. Have a team member position themselves aside of the car which is positioned slightly behind the starting line and push the car towards the CBR®. Be sure that

when the car crosses the starting line the only force acting on the car is the frictional force.

6. Note the distance of the car from the CBR® when it comes to rest. You do not need to make a graph of Distance vs. Time for this since you only need to subtract this distance from the distance of the CBR® from the starting line. After all, the accident investigators are seldom witnesses to the accident and hardly ever recording the data. They only know where the skid marks started and where they ended.

7. Use the Velocity vs Distance graph for the car and the trace option to determine the speed of the car in ft./sec. at the time it crossed the starting line.

8. (Optional) Have a contest with your team members or all of the members of the class to see who can come closest to pushing the car so that it crosses the starting line with a speed of 3 ft/sec (≈ 2 mph).

PROCEDURE 1 Alternate − Requires Some Trigonometry

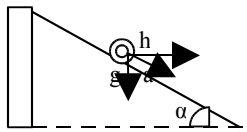
Determining the Coefficient of Friction

1. On the floor of your classroom or on a lab table set up the experiment as indicated in the photo at the end of the Background Information section.

2. Measure the following distances:
   a. The height of the platform on which the top edge of the incline is resting.
   b. The horizontal distance of the bottom edge of the incline from the top edge.
   c. The length of the incline.

3. Using right angle trigonometry and your measurements, find the sine, cosine, and tangent of the angle that the incline makes with your lab table or the floor.

4. Below is the force diagram for the car moving down the incline:



Note that for this experiment the horizontal force, h, is zero for this experiment. Use a value for g or 9.88 m/sec^2 to find the value for the acceleration along the incline, a.

5. The following relationships are based on Newton's Laws of Motion:

$$v = a*t + v_0$$
$$d = \frac{1}{2} a*t^2 + v_0*t + s_0$$

Where t = time since the car was released; v = velocity of the car at time, t; d = distance traveled by the car at time, t; $v_0 = 0$ = the initial velocity of the car; and $s_0 = 0$ = the initial distance traveled by the car. Find the time at which the distance is equal to the length of the ramp and then find the velocity at that time. This is the actual velocity of the car and is not subject to a sampling error, assuming that your measurements in step 4 are accurate.

6. Release the car as illustrated below.

7. Measure the distance from the end of the incline to the back wheels of the car. This is the distance traveled by your car when the only force affecting its motion is the frictional force.

8. Using the calculated value of the velocity of the car at the base of the incline from step 5 and the distance traveled by the car from step 7, find the value for the coefficient of friction using formula (1) given in the **Background Information**. Write this value down since you will be using it in the next part of the experiment.

9. Using this value for the coefficient of friction and Derive® draw a graph of starting Velocity vs Distance for the car when it is moving only under the influence of the frictional force. Allow for distances up to six feet in length.

Using Your Findings to Determine Starting Velocities

1. Using a piece of tape at least 30 cm. long, or a straight edge and a graphite pencil, draw a "starting line" on your lab table or on the floor where you were working for the first part of this activity.

2. Have a team member position themselves aside of the car which is positioned slightly behind the starting line and push the car in a direction perpendicular to the starting line. Be sure that when the car crosses the starting line the only force acting on the car is the frictional force.

3. Note the distance that the car is when it comes to a stop. After all, the accident investigators are seldom witnesses to the accident and hardly ever recording the exact velocity data. They only know where the skid marks started and where they ended.

4. Use the Velocity vs Distance graph that you drew for the car and the trace option to determine the speed of the car in ft./sec. at the time it crossed the starting line.

5. (Optional) Have a contest with your team members or all of the members of the class to see who can come closest to pushing the car so that it crosses the starting line with a speed of 1m/sec (≈ 3.6km/hour).

Returning to the Accident in the Scenario

The police on the scene of the accident determined that the roll over of the car meant that the car was traveling at a speed of at least 50km/hour when it first started to roll. The skid marks prior to the roll over were 55m long. Using the standard tables for the type of road surface and the existing climatic conditions they used a value of .7 for the coefficient of friction, *f*.

1. Use Derive® and equation (1) draw a graph for velocity versus distance using the police's value for the coefficient of friction.

2. Determine the distance that a car traveling 50km per hour would skid.

3. Add this distance to the measured 55m of the skid and estimate a minimum speed that the vehicle was traveling prior to applying the brakes.

Coroner's Comments

The calculated speed of this vehicle will amaze you. It does not seem to be too excessive. However, it was enough to cause the vehicle to skid 55' across the road and still have enough velocity to force the car to roll. The major cause of the accident was that the driver was speeding when he came into the curve that preceded the stretch of road shown in the photo. Because the driver's judgment was impaired and he was an inexperienced driver, he lost control of the car on the curve and overcorrected by applying the brakes. This is not a rare occurrence and unfortunately happens far too often. Drinking and speed increase the risk of this type of accident. It is not likely that this victim would have survived even with a seatbelt but with a seatbelt he would not have been ejected from the vehicle. A slower speed would have greatly increased his chance for survival. The only 'good' thing was that there were no other vehicles involved.

Name:_____

Date: _____

Alternate

Analyzing Motor Vehicle Accidents

Procedure 1

DATA RECORD

Description of Feature	Measurement
Height of Incline (cm)	
Length of Surface (cm)	
Length of Base of the Incline (cm)	
Sine of the Angle at the Base of the Incline	
Acceleration of Car Down the Incline (m/sec^2)	
Calculated Velocity of car at end of the incline (m/sec)	
Distance traveled after the incline (cm)	

Finding the coefficient of friction

1. Illustrate your calculation for the velocity of the car at the end of the incline.

2. Use the information in your Data Record and equation (1) to calculate the coefficient of friction for your car. Show the steps of the calculation.

Determining Starting Velocities

DATA RECORD

Coefficient of Friction:

Graph of Velocity vs. Distance Traveled from Equation (1):

Trial #	Distance Traveled from Start	Calculated Velocity at Start (use graph)
1		
2		
3		
4		
5		

Graph of velocity vs distance using the Police Table value for the coefficient of friction.

TEACHER NOTES

CONCEPTS COVERED IN THIS ACTIVITY

The main concept under investigation here is the connection between skid mark length, which is really the effect of frictional force, and the speed of the car. The connection is made as a result of two equations for kinetic energy. If this is a concept that has not been covered for your students, it is possible to explain this relation on a heuristic level. Kinetic energy is the energy related to motion. Most students will understand the connection between the amount of energy required to keep a car traveling at a certain velocity. The higher the velocity, the more energy required. This, of course, does not explain the squared relationship with the velocity. On the other hand, that can be covered later in the student's career. The relationship between frictional force and kinetic energy can be understood by sliding a load over different surfaces or sliding the load and then placing a swatch of carpet under the load and sliding it again. Students can feel the effect of reducing the friction. The main issue is that students have a feeling for the basic equation for this activity, Equation (1):

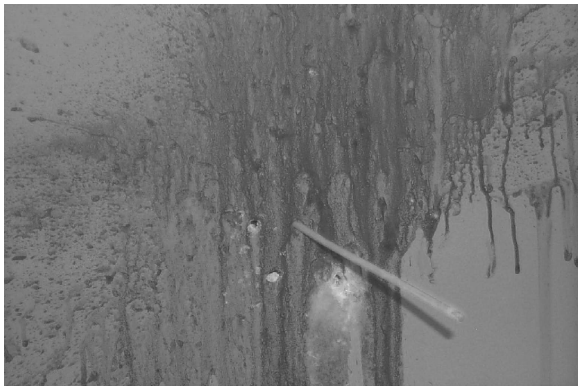
$$v = \sqrt{2g \cdot f \cdot d} \quad .$$

Many students will be surprised that the weight of the vehicle does not appear in this equation. This is a result of equating the two expressions for kinetic energy. This gives an excellent opportunity to mention Gallileo's classic experiment that showed that the speed of a falling body is dependent only on the force of gravity, not the weight of the object. You can also point out that the equation does not ignore the effect of gravity since the gravitational constant, g, appears in the equation. A skid on the moon at the same velocity would be a lot longer if the coefficient of friction is the same as here on earth.

It is far too dangerous to have students produce legitimate skid marks or even to go out on the highway in search of skid marks and investigate them. What is done in this activity is to replace the friction of unmoving tires by the friction of the axel on the guide of a toy car. No rubber on the road, but the basically the same forces at work. Your choice on whether to use Procedure 1 or Procedure 1 Alternate depends partly on your preference and comfort with using the CBR® and partly on whether or not your students have had an introduction to Trigonometry. If your students have had the introduction to Trigonometry and you can afford the time, it is interesting to do both procedures. The alternate procedure gives the exact value for the speed of the vehicle at the end of the incline. The first procedure is subject to some sampling error as a result of the fact that the CBR® can only sample in intervals of at least .05 seconds.

SPITTER SPATTER

Analysis of Blood Spatter Patterns



SCENARIO

As an investigator for the coroner's office, you have been called to the scene of a homicide/suicide. The residence is a one story ranch house with bedrooms on opposite ends. The owners of the residence were in one bedroom and their daughter's bedroom is on the opposite end of the house. The parents heard nothing during the night and became suspicious when they attempted to awaken the daughter to go to work and found the bedroom door locked. When there was no answer they forced the door open. Inside the bedroom they found their daughter lying on her right side with multiple gunshot wounds of the head. Lying next to the daughter was a young man with a single gunshot wound of the head. Multiple holes are found in the walls in front of the daughter's face along with massive blood spatter. Because of the massive amount of trauma it will be necessary for you to determine the angle of the shots as well as the distance from the shooter.

BACKGROUND INFORMATION

In many criminal investigations that involve a death resulting from and external injury such as a gun shot, blow with a blunt object, or knife wound, it is crucial to the investigation to know the position of the victim or assailant. In this way the investigator can tell whether the victim has been moved or the scene has been changed in any way. Sometimes the only evidence visible is the blood spatter on the surrounding surfaces. With the use of knowledge from physics and mathematics the investigator can use this evidence to determine the point of origin for the spatter.

The main physical principle is derived from the fact that blood is a very thick or viscous liquid. As a drop of blood flies through the air it naturally assumes a shape that minimizes its surface area. This shape is that of a sphere However, when the blood drop hits a flat surface, such as a wall or floor, the three dimensional shape becomes a two dimensional shape. If it strikes the surface at a 90 degree angle, it expands equally in all directions to its minimizing area that will contain the volume of the drop, a circle . However if the sphere hits at an angle, the bottom of the sphere hits the wall while the top continues moving. This is the situation shown in Figure 1.

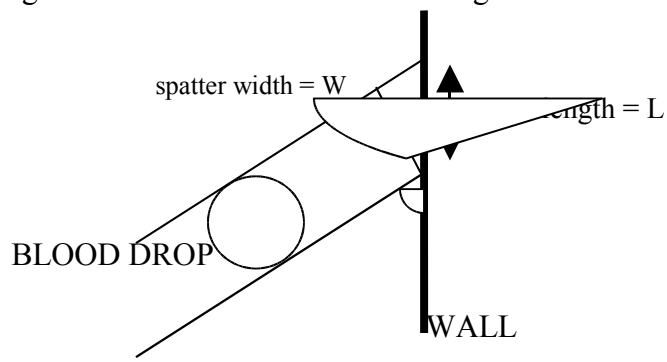


Figure 1: The result of the blood drop hitting a flat hard surface

Note that the width of the spatter is the same as the width of the original droplet of blood. The amazing fact is that the angle at which the spatter hit the wall can be determined from the spatter width and the spatter length on the wall.

The shape of the spatter resulting from drop hitting the wall is a teardrop made up of an ellipse and a slight extension resulting from the momentum of the flight.

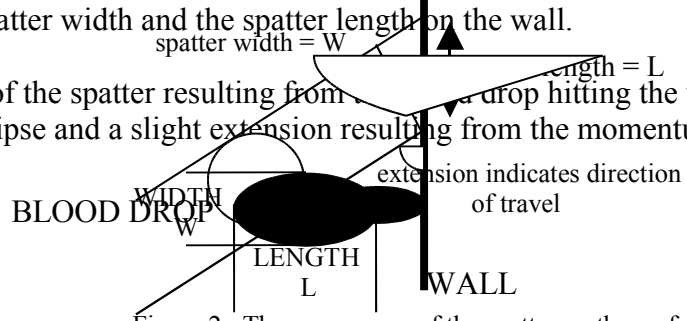


Figure 2: The appearance of the spatter on the surface

The long axis of the ellipse points in the direction from which the blood was dispersed across the surface. Thus, taking several spatters and extending them along the long axes of the ellipses made by the drops in the direction opposite to their extensions, the lines should converge (intersect) in one general area on the wall.

But the point of convergence on the wall does not tell how far the victim was from the wall. A study of Figure 1 will tell us how we can determine the position of the source of the spatter. The angel, α, is the angle at which the spatter hits the wall. From elementary right angle trigonometry we can derive relationship (1).

$$\sin(\alpha) = \frac{W}{L} \qquad (1)$$

NOTE: The value, L, does not include the extension of the spatter shown in Figure 2. In order to determine the value for L start at the widest portion of the spatter and measure to the end opposite the extension. This will give half of the length of the long axis or ½ L.

From (1) we can find α, using the ARCSIN( ) function in Derive® . This will give the angle in degrees.

$$\alpha = \sin^{-1}\left(\frac{W}{L}\right) \qquad (2)$$

In order to find the origin of the blood drops that formed the spatter, a thin thread is attached to the center of the individual spatter droplet and placed in the direction of the line extending the long axis of the ellipse. Raise the thread until it makes an angle of α with the long axis. Continue extending the thread maintaining this angle until it is in front of the point of convergence and this is the position where the blood spatter originated.

This activity is in two parts. The parts do not depend on each other. In the first part you will create a "virtual spatter pattern" on the screen of your calculator and find the line that will lead to point of origin. Since you will only be working with one spatter pattern, it will be impossible to tell where along this line the point of origin is located.

The second part of the activity uses your calculator only to evaluate formulas. You will create an actual spatter pattern using fake blood and a paint brush. This procedure is rather messy and your school may not have the facilities to accommodate such a procedure. After spraying the blood drops by flicking a wet paint brush, you will choose several spatter patterns, analyze them to determine the point of convergence and the angles of impact. From this information you will determine the point of origin for the spatter pattern.

FORENSIC OBJECTIVES

- To determine the point of origin for a blood spatter pattern.
- To determine if there are multiple points of origin or only one point of origin for the pattern.

SCIENTIFIC/MATHEMATICS OBJECTIVES

- To learn how to use the definition of an ellipse to construct an ellipse with its long axis lying in a given direction.
- To use the information obtained by determining the long axes of the ellipses making up several spatter patterns to determine a horizontal point of convergence.
- To use the ratio of the short axis of the ellipse of a blood spatter to the long axis to determine the angle at which the blood droplet creating the spatter struck the wall.

MATERIALS

Procedure One

- Derive
- A piece of black cotton thread about an arms length long.
- A protractor
- A meter stick

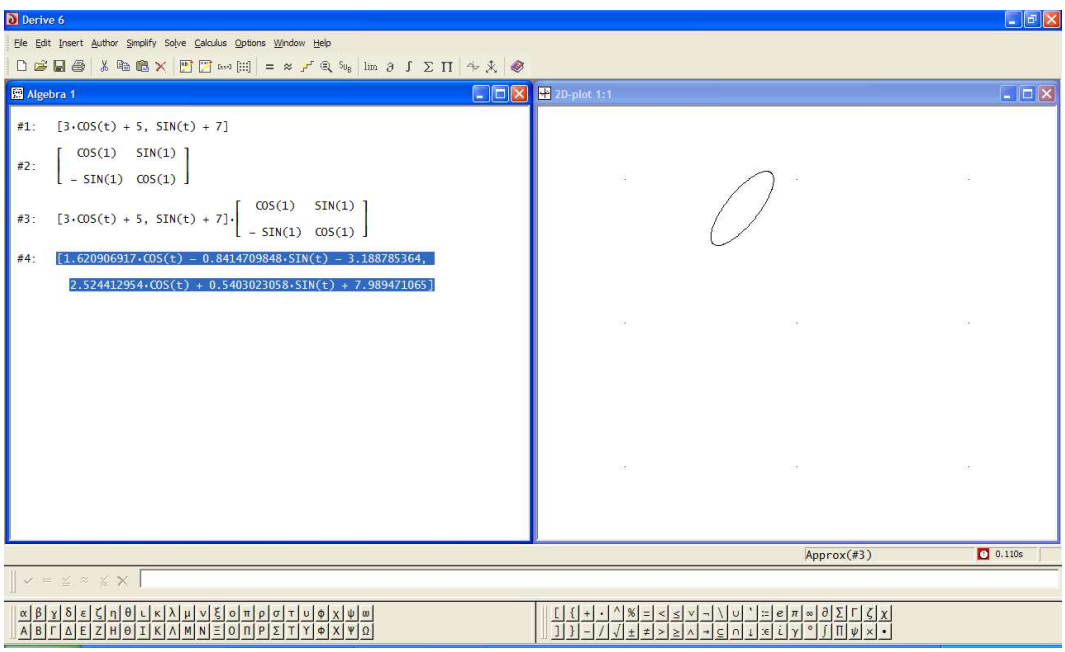
Procedure Two (in addition to the above – CabriJr® not required)

- A piece of paper from an artist's sketch tablet (12 inches by 18 inches)
- A tooth brush
- A glass of water dyed red using food coloring
- Tape or tacks to hang the plastic and sketch paper

PROCEDURE ONE

CONSTRUCTION OF A "VIRTUAL BLOOD SPATTER"

1. Using Derive® draw the graph of an ellipse that is rotated by some angle. The following screen shot shows a method using the parametric form of an ellipse. Use any method that you are comfortable using.



2. The graph will be your "blood spatter". You will use it to determine the angle that the spatter hit your screen.

3. Measure the length of the long axis of the ellipse, L, and the width of the ellipse, W. From these two measurements and equation (2), determine the angle at which the spatter hit your screen.

4. Hold the black thread at the center of the ellipse and extend it along the long axis of the ellipse at an angle of α with your computer screen. This will give you the direction from which the droplet came that made the spatter.

5. Pick some point along the thread and have a friend use their thread to go to another point on your screen.

6. Challenge your friend to draw a spatter centered at the point on the screen where their thread touched the screen, with a long axis in the direction of the thread and angle equal to the angle that the thread met the screen.

PROCEEDURE TWO

Analyzing A Complete Blood Spatter

1. Carefully secure the piece of plastic sheeting that you were issued to the wall.

2. Place the sheet of artist's sketch pad paper in the middle of the sheeting and at least six feet above the floor. You may need a ladder to do this.

3. Make sure that you are wearing your lab goggles and artist's smock. You need to protect your eyes and your clothing.

4. Position yourself about 5 feet away from the wall and off to the side of the paper a few feet. Make sure no people or objects are between your position and the paper. In fact, it is a good idea to have everyone stand behind you.

5. Lightly dip the end of your tooth brush into the "blood" mixture.

6. Hold the brush with the bristles pointing towards the paper at about the level of your shoulder and aiming at the paper, flick your wrist with enough force to propel the mixture on the end of the brush towards the paper.

7. Examine the paper to be sure that you have at least three or four good spatter patterns on the paper. If this is not the case, remove the paper and try again until you have a good pattern. If after several tries you still do not have a good pattern, you may need to thicken your mixture with some corn starch, but not so much that you no longer have a liquid.

8. When you have a good sample, allow it to dry for a few minutes and then take it down and carefully carry it to your class room.

9. Using four or five of the good patterns, follow the long axes of the ellipses to determine the area of convergence relative to your sheet. It is unlikely that all of the lines will meet in exactly one point, but they should be within a fairly close proximity of each other.

10. Now choose one of the best spatters and carefully measure (you may want to use a compass to get more accurate measurements) the length of the short and long axes of the ellipse. NOTE it does not matter what units you use for these measurements since the ratio of the short axis to the long axis is all that matters.

11. Find the angle at which the droplet hit the paper using your calculator.

12. Fasten the sheet of paper to the wall of your classroom and locate the point of convergence of the extended long axes of the spatters.

13. Fasten a thread to the center of the spatter you used to determine the angle.

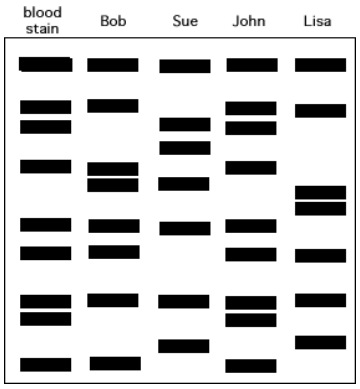
14. Have one team member hold the protractor and make sure the thread remains at the proper angle as you extend it until it is in front of the point of convergence.

15. When you reach the point of convergence, compare the location of the thread to the position of the paint brush when you flicked your wrist to create the spatters.

CORONER'S COMMENTS

The coroner's role in this scenario is vital to the investigation. Close examination of the wounds will tell you how close the weapon was to the victim when it was fired. X-rays will assist you is making the determination of the angle as well. Placing wooden dowels in the holes in the wall will also determine the angle of each shot. The temperature probe, as well as the vitreous humor potassium levels, will assist you in finding the time of the incident. Complete interviews with the family will give you the entire picture of the surrounding circumstances. You will have found that the two victims in the scenario had a relationship and the daughter had recently informed the young man that she wanted to end that relationship. The police will find the spent projectiles and compare them with the evidence obtained from the weapon that was found on the scene.

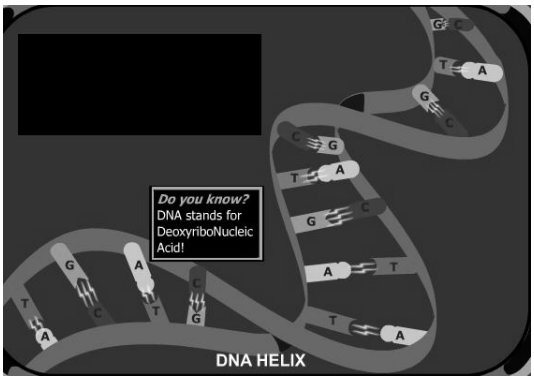
Activity
IT'S ALL RELATIVES
The Role of DNA Evidence in Forensic Investigations



SCENARIO

You have responded, as a result of a call from the police to the Coroner's Office, to the scene of the death of a victim who was mugged along a dark street. You arrive and find the victim lying on his back with severe blunt force trauma (injury) to his head and arms. There is blood on the sidewalk. When examining this victim you take notice that there are injuries to both hands which may indicate that the victim was fighting back as well as warding off blows to his body. You are instructed by the pathologist to place paper bags over the victim's hands and seal them using twine at the victims wrists. You are also instructed to provide a sample of the blood on the sidewalk. When the autopsy is performed, the pathologist finds that there appears to be tissue and blood under the finger nails of the victim. It is sent to the lab for analysis and DNA testing. The victim's blood is also tested to rule out that it was his own blood. The blood sample from the sidewalk matches the victim's blood.

BACKGROUND



DNA stands for Deoxyribose Nucleic Acid. The structure of DNA was discovered in the early 1950's by J. D. Watson and F. H. C. Crick and announced in a one page article in 1953. DNA contains the basic blue print for life and the function of various organisms. It has the shape of a twisted ladder, or the famous double helix. The "rungs" of this ladder are weak hydrogen bonds between nucleic acid molecules. There are four bases that distribute themselves along the ladder at each bond: Adenine, Cytosine, Guanine, and Thymine. Denoted by a four letter DNA alphabet: A, C, G, and T. The bases on either side of the ladder's rungs are joined in a very specific manner. A always pairs with T, G always pairs with C. These pairings are called, not surprisingly, base pairs (short hand – bp's).

Here is an example of a DNA strand as it is reported to the researcher, AGTCTCGAATAAATGCTGAAGGTA…. It is a "word" over a four letter DNA alphabet. On the other side of the ladder is found TCAGAGCTTATTTACGTTCCAT, the complementary word for the example word.

The Human Genome, the complete listing of our DNA is over 3 billion base pairs long. It is found in the nucleus of each of our cells. However, only 0.1% is unique to any individual. That still leaves over 3 million base pairs to be compared with samples from each suspect. Furthermore this unique DNA is spread over the complete genome. Fortunately, there is another source in the cell for our DNA.

Also contained in the cell are self replicating organelles called mitochondria. They contain DNA, but with a much shorter genome of only 16,569 base pairs or 1/300,000 the length of nucleic DNA. It contains only 13 sites that are unique to each individual and enable the individual to be identified. This makes mitochondrial DNA easier to analyze and match in the forensics lab. The probability of a random matching of two strands of mitochondrial DNA is less than one in $3.4*10^{9,975}$ for individuals other than identical twins. Can you identify who may be the "culprit" from the illustration of the site comparisons at the beginning of the Activity?

Here are some of the **forensic uses** of DNA.

- Identify potential suspects whose DNA matches evidence left at crime scenes
- Exonerate persons wrongly accused of crimes (the most common use).
- Identify crime and catastrophe victims
- Establish paternity and other family relationships. (must use nucleic DNA probably from the Y-chromosome)
- Identify wildlife species in a case involving poaching.

FORENSIC OBJECTIVES

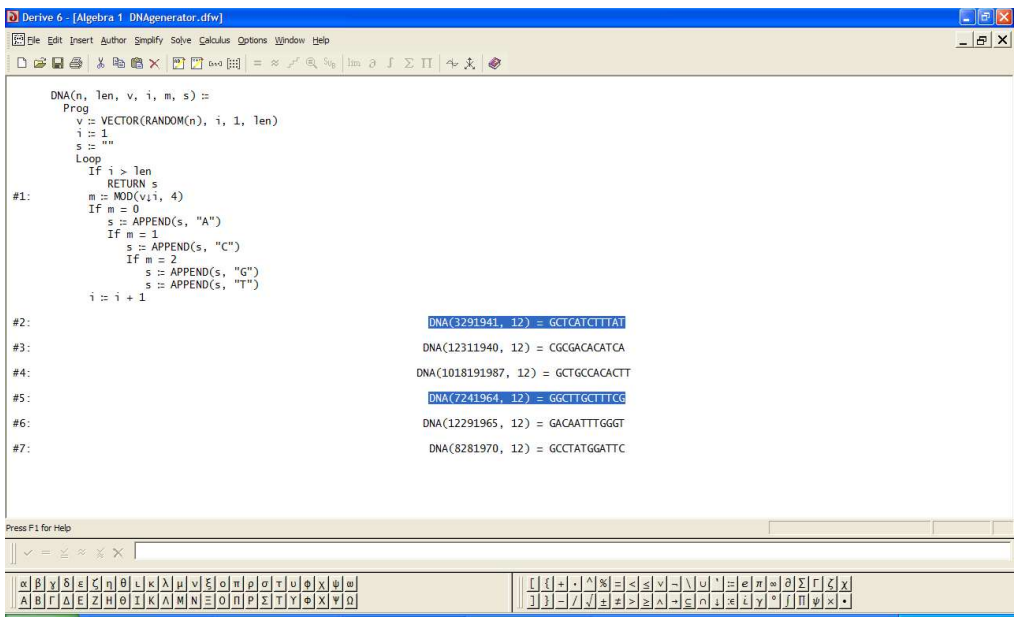
- To provide an introduction to the use of DNA evidence in determining the likelihood that a suspect committed a crime.

- To learn a technique for aligning DNA samples for the purpose of comparing them.

PROCEDURE

Making Your Personal "DNA Sample"

1. Since do not have time to process samples of blood or saliva, we will generate a small string of DNA that is related to you. We will use the function, `DNA(n,len)` shown in the following screen shot:



   In this function, *n* is the seed for the random number generator and *len* is the length of the "DNA" sequence we are generating. In order, to personalize your string, set *n* = ddmmyyyy where dd is the day of the month you were born (if it is a single digit, there is no need for a leading zero), mm is the month you were born (use a leading zero if necessary) and yyyy is the year that you were born, i.e. it is your birth date. The length of the string for this exercise will be 12. So your function call will be `DNA(n,12) =`.

2. Write down your sequence.

3. Record your name (or ID number) on a master list.

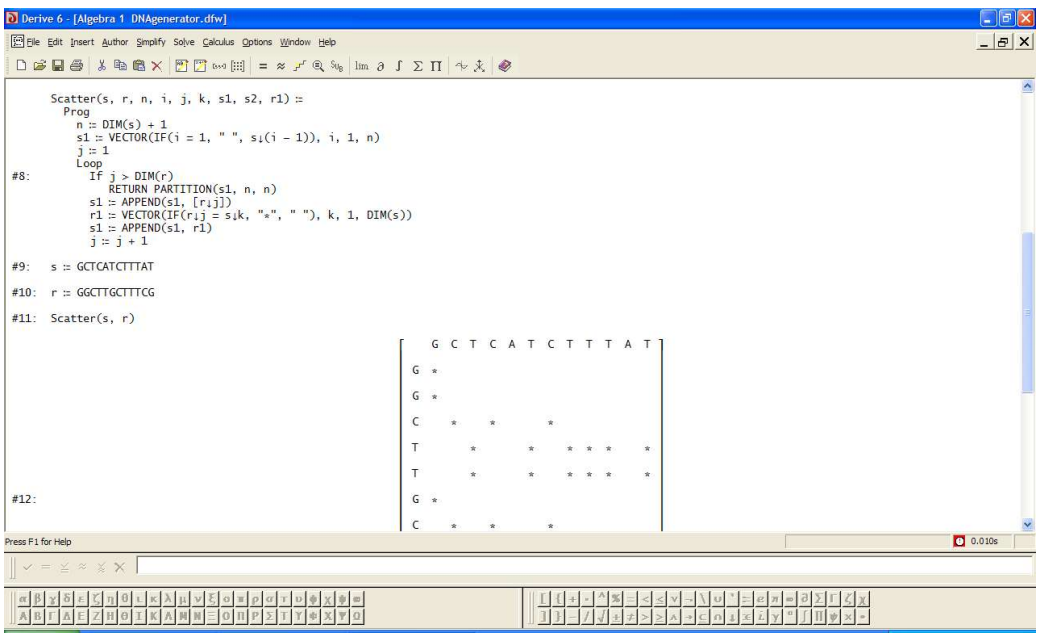
4. Given that for each position in the list the probability of two sequences have the same letter is ¼ , what is the probability that two independently generated

sequences will be the same? What is the probability that two sequences in your classroom will be the same?

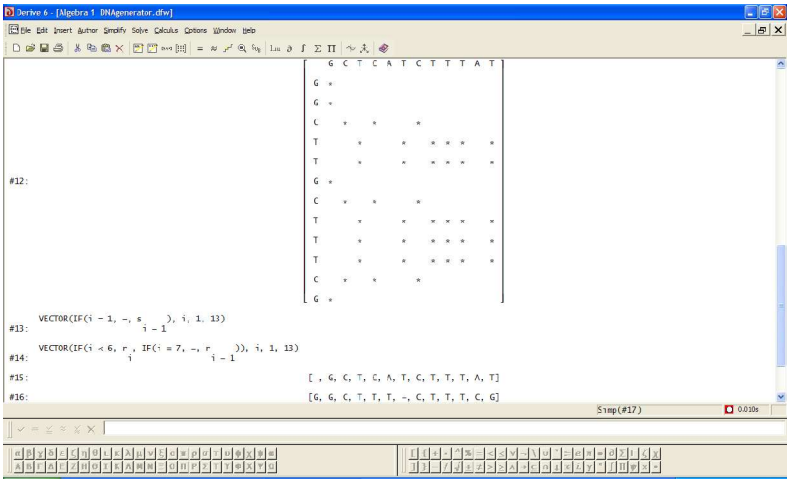
Aligning Two DNA Strands for Comparison

1. After having viewed the class "DNA Pool" choose a string from the data that seems to be "close" to your string. (In the unlikely event that there is a string identical to your string, choose a string that is not identical to yours.) NOTE: close is a loosely defined term. In the above screen shot, I chose the sequences: GCTCATCTTTAT and GGCTTGCTTTCG.

2. We create a 13 X 13 matrix that has a blank in the upper left hand corner; the (1, 1) position. Across the remainder of the top row place the elements of the first string, s. Down the remainder of the first column place the elements of the second string, r. If i >1 and j>1, place a blank in the cell if $s_{i-1} \neq r_{j-1}$ and a * in the cell if $s_{i-1} = r_{j-1}$. The resulting matrix is called a *Scatter Plot*. The program to create a scatter plot is shown in the following screen shot.



3. When examining a Scatter Plot you look for diagonals with a sequence of *'s on them, usually three or more. This will provide a guide for aligning the sequences. Up until the early 1990's Scatter Plots were one of the main tools for aligning two sequences of DNA. The next screen shot shows you the complete scatter plot for the two sequences chosen in step 1.



4. An examination of the Scatter Plot shows that positions 2, 3, and, 4 of sequence, r, match positions 1, 2, and 3 of sequence, s. We place a "-" at the beginning of sequence s. This moves everything over one position and now the three nucleotides line up. Now positions 7, 8, 9, and 10 of sequence, r, match positions 8, 9, 10, and 11 of sequence, s. We place a "-" between positions 6 and 7 of sequence, r, and these positions are aligned. The "-" signifies we are hypothesizing that in one of the sequences a nucleotide may have been inserted or deleted during the DNA replication process.

5. Use your sequence and the one you choose as a close match and create a scatter plot. Using the plot, create the alignment that you think may be the best one for the two sequences.

CORONER'S COMMENTS

Finding tissue and blood under the nails is extremely important evidence. It is especially important when a violent crime has been committed. The DNA may rule out or quickly point to the appropriate suspect. There are several 'proverbs' about evidence. The first is that is that anyone coming into or out of a scene either leaves something or takes something away. The second is that evidence just lies there and waits for someone to find it. Taking the time to properly process a crime scene can well make the difference in solving or not solving a crime. DNA typing has been one of the most important crime solving tools that we have today. Blood is not the only substance used for DNA typing. Any body cell or fluid can be tested i.e. skin, saliva, semen or hair.

Directing Our Suspicions

Using Matrix Algebra to Organize the Results of an Investigation

Scenario

You are called by the county police to investigate a shallow grave that has been located by two hunters at the edge of the county. You cordon off the scene and carefully uncover the remains that are buried in the grave. It is obvious that the body was not buried recently. In fact all that remains is essentially bones and some shreds of fabric. It is apparent to your experienced eye that the victim was a young female, but that is all that you can determine at the scene of the grave. You assist in a careful exhumation of the body and direct that it be taken to the county morgue.

Returning to the morgue, you do a more precise analysis. From the length of the bones and an examination of the bone ends you determine that the victim is a pre-pubescent female. A forensic dental examination reveals that the victim is a young girl who was reported missing over 20 years ago. Further analysis reveals that the victim was shot at least three times in the area of the chest. No bullets have been found on the scene, but the extent of damage to the bones indicates that the weapon was on the order of a .45 caliber pistol. You are unable to make a more exact estimate concerning the weapon.

At this point your involvement in the investigation is over. Solving the crime is the job of the police. A team of detectives is assigned to the case and they begin the tedious job of interviewing people who knew the victim and also others who had contact with her. The town where the victim lived is a small town, so the detectives begin there. They also review the local police's case notes. The prime suspect at the time of the disappearance was an African-American 14-year old boy who was the son of the girl's neighbors. His family was the only non Caucasian family in the neighborhood. The boy's sister was the girls "best friend"; however, the victim and his sister had just had a small falling out that had seemed to have been orchestrated by the victim's mother.

The Police Investigation

The police began by questioning the now grown boy and his sister. As is the case in so many investigations, the questioning revealed other individuals who needed to be visited. From their interviews, the police have narrowed the suspect list down to six individuals. Here is a very brief summary on the police findings. We will identify the suspects by alphabetical letters, A through F.

A. The African-American boy. Sister, of whom he is very protective, had just received a "rejection" from the victim; Had been seen on the victim's property the night of the disappearance; Determined to have been a fairly normal teen age boy, but resentful of the discrimination his family had to endure.
B. African-American girl. Had been a friend of the victim from the moment her family moved next door to the victim's family amidst a demonstration by the local "citizen's committee for a pure neighborhood"; Had recently had a childish falling out with the victim; Had written a note to the victim the day of her disappearance asking to meet in their "secret place."
C. Leader of the citizen's committee that lead a demonstration greeting the moving van when the African-American family moved in; Was very offended by the victim being good friend and confidant of suspect B; Quoted as saying "The only thing worse than a black is a white person who befriends one"; Known to be a stern father and husband. Known to have kept a gun in his home.
D. Real Estate agent trying to capitalize on the panic that can result from having a non Caucasian family move into the neighborhood; Was trying to buy homes cheap and sell houses in his new development on the edge of town; Perceived the public friendship of the victim and suspect B as "bad for business."
E. Teen aged boy. Son of suspect C; Prejudiced, but confused, especially by the friendship of the victim and suspect B; Had been publicly rebuked and slapped by the victim when bullying her at school; Apologized to the victim and her friend and was, in return, severely disciplined by his father. He was constantly trying to please his father.
F. Mother of victim. Was entranced by suspect D's attempts to sell her a new home, but opposed by her husband; Insisted that her daughter stop seeing suspect B; Made her daughter ignore her friend in public.

Each of the above suspects seems to have a reason for being placed on the list of suspects to the crime. The job for the detectives is to sort out these reasons and determine on whom to concentrate their efforts.

An old rule of thumb in criminal investigations is to look at three pointers: Motive, Opportunity, and Profit. Who had the most powerful motive for killing the victim? Who had the best opportunity to commit the crime? Who would profit the most from the crime? We will abbreviate these pointers as MOP. In the following investigation, we will apply the MOP pointers to the crime at hand.

Investigative Objective

- Organize investigative findings in a sensible way that will lead to the solution of the crime.

Mathematical Objectives

- Find a way to sensibly incorporate subjective evidence into a meaningful quantitative analysis.
- Assign a priority ranking and strength to the various alternatives for solving the crime.
- Using a two way comparison of alternatives to achieve the ranking.
- Learn about the concept of a dominant vector related to certain matrices.

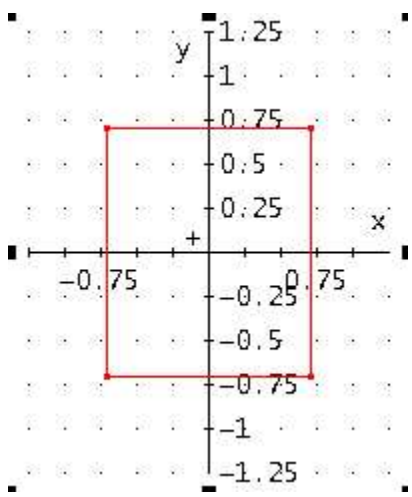
Preliminary Mathematical Discussion

In this section we will introduce the idea of a dominant vector which can be associated with certain types of matrices. At the moment, this may seem like an irrelevant mathematical diversion, but it really has a lot to do with our attempt to solve the crime; however, the idea of a dominant vector (more properly called a dominant eigenvector) is the basis of the *Analytic Hierarchy Process* invented by the mathematician Thomas Saaty in the late 1970's. This process is a way of prioritizing several alternatives into a logical sequence of choices.

We will illustrate the idea of a dominant vector while working with a 2X2 matrix. We will assume that every entry in the matrix is positive and that the second row is not a multiple of the first row. For our example matrix we will use the matrix,

$$A = \begin{bmatrix} 7 & 8 \\ 1 & 2 \end{bmatrix}$$

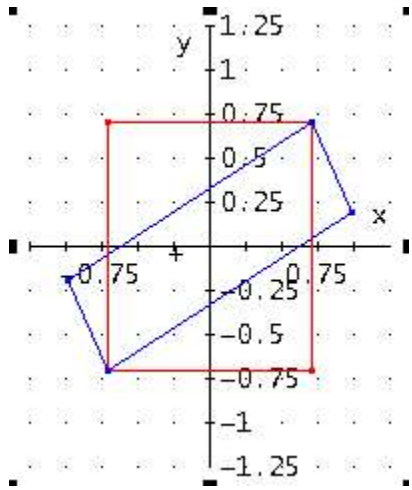
We will look at the effect of multiplying the corners of the following square by the matrix, A.



Since multiplication by the matrix A is a linear process, the lines joining the corners will be transformed into lines joining the corners of the images of the corners. Also any point in the interior of the square will be transformed into the interior of the image of the square. The square can be created by joining the following coordinate points in order:

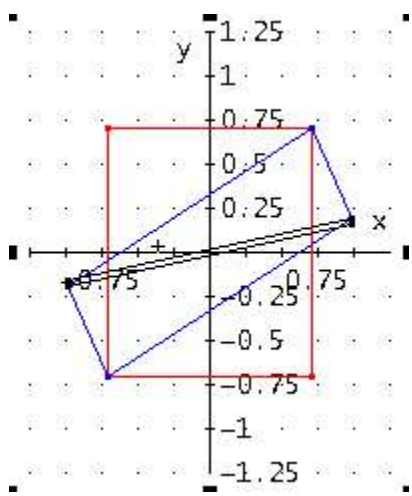
$$\left[\frac{-\sqrt{2}}{2},\frac{-\sqrt{2}}{2}\right],\left[\frac{-\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right],\left[\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right],\left[\frac{\sqrt{2}}{2},\frac{-\sqrt{2}}{2}\right],\left[\frac{-\sqrt{2}}{2},\frac{-\sqrt{2}}{2}\right]$$

We multiply each of the points by A and normalize by dividing by the length of the resulting vector.

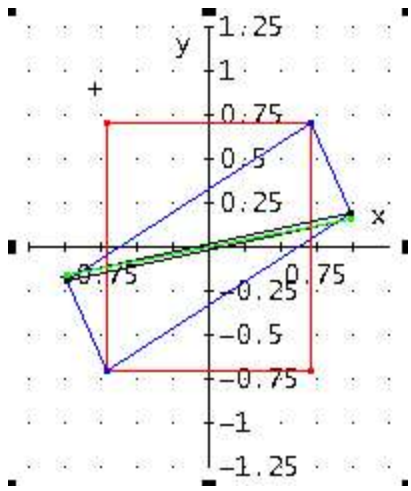


The result is shown as the blue rectangle superimposed on the original red square. Note that the rectangle is now slanted in a particular direction and is a compressed form of the original,

Next multiplying the corner points of the blue rectangle by A or doing the same thing by multiplying the corner points of the red matrix by A^2, we see a rectangle with a slightly different slant and even more compressed. This rectangle is shown as a green rectangle.



Repeating the process one more time we see:



At this stage we have almost reduced the square to a single line. In fact, if we continue taking higher powers of A times the coordinates points of the square, we see that the square is reduced to a straight line segment lying on the line

$$y = 0.15936x$$

which passes through the origin and the point, **d** = (0.8625413912, 0.1374586087). As an exercise, take an arbitrary point in 2-dimensional space and multiply it by a high power (20 or higher) of the matrix, A, and show that the result lies on this line (or very close to it).

We choose the vector **d** lying on the line and having the property that its components sum to 1, and call it the **dominant vector for the matrix, A**. The idea of a **dominant vector** will play an important role in our analysis of the evidence collected by the police during their investigation of the suspects in the murder.

It is important to note that not every matrix has a dominant vector. For example, the matrix

$$B = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Has the property that when any vector is multiplied by successive powers of B the result is a set of 12 different vectors that keeps repeating.

Fortunately for us, we will not be dealing with matrices like B. Instead we will be dealing with matrices that have all positive entries. Two German Mathematicians, Oskar Perron and Ferdinand Georg Frobenius, proved that *every matrix having all positive entries has an associated* ***dominant vector*** *having all positive terms*. They actually proved much stronger results, but the statement we have given in this paragraph is sufficient for what follows.

Applying Hierarchical Ranking Methods to the Evidence

The Analytical Hierarchical Ranking Method was devised by Thomas Saaty in 1977. It is a method for ordering preferences of several categories or individuals relative to several criteria. In our case we have six suspects, A through E, and three criteria, Motive, Opportunity, and Profit. This technique is especially useful when we are faced with a situation that is hard to quantify, but where we have opinions or feelings when comparing the categories, individuals, or criteria, and we can rank and rate the relative strength of our feelings. In effect, the method gives us a way to quantify some things that seem to be non quantifiable.

First we need to have a scheme for comparing two items relative to a particular criterion. The method devised by Saaty uses a scale of 1 to 9 for comparing the strength of feeling about two individuals relative to the criterion.

1 = There is **no preference** for one item over the other
3 = A **slightly stronger** preference for one over the other
5 = There is an **essential** preference for one over the other
7 = There is **some evidence** that one is preferred over the other
9 = There is **indisputable evidence** for preferring one over the other

2, 4, 6, 8 are intermediate values used as compromises for disagreements in group decisions. In short, this is the standard preference scale of 1 to 5 expanded to provide intermediate values for compromises. The idea is to store preferences when comparing two suspects in an $n \times n$ matrix $[a_{i,j}]$, where

$a_{i,j} = s$ where s is the score of the preference of item i over item j

$$a_{j,i} = \frac{1}{s}$$

This implies that we always have $a_{i,i} = 1$ for all $i = 1, \ldots, n$.

Let's try out the scheme for suspect A relative to the other suspects. We will assume that a team of at least four detectives was assigned to the case and they based their scores on their interviews.

A compared with B

The detectives found no strong evidence to rank one over the other relative to the motive criteria, so they gave a score of 1. Thus,

$a_{1,2} = 1$ and $a_{2,1} = 1$

A compared to C

Although suspect C wanted the African-American family to move out of the neighborhood, the detectives believed that the hurt suffered by A upon rejection by her former friend was a stronger motive so they gave a score of 7.

$$a_{1,3} = 7 \quad \text{and} \quad a_{3,1} = \frac{1}{7}$$

A compared to D

The victim was actively standing in the way of the Real Estate Agent's sales perspectives in the neighborhood for his housing project as well as the opportunity to pick up some valuable property cheaply. They all believed this was a stronger motive than A's hurt, although this was a feeling not supported by evidence so they rated it as a 5.

$$a_{1,4} = \frac{1}{5} \quad \text{and} \quad a_{4,1} = 7$$

A compared to E

The detectives were split on this comparison. One group believed that the public embarrassment of E by being slapped by the victim in public and his subsequent criticism by his father was clearly stronger motivation to kill the victim than A's rejection by her friend. The other group agreed that it was stronger, but not quite as strong as the first group perceived it to be. The compromise score of 8 was chosen.

$$a_{1,5} = \frac{1}{8} \quad \text{and} \quad a_{5,1} = 8$$

A compared to F

The consensus was that the rejection that A experienced was a slightly stronger motive than the mother being thwarted in her desire for her dream house by her daughter's sway with the father not to move. They gave a score of 3 to A.

$$a_{1,6} = 3 \quad \text{and} \quad a_{6,1} = \frac{1}{3}$$

The above assignment of values has filled in row 1 and column 1 of the Motive matrix. A similar analysis was done for the remaining suspects. Note that there are 10 more comparisons to be made in order to fill out the matrix. Here is the matrix proposed by the detectives:

$$Motive = \begin{bmatrix} 1 & 1 & 7 & \frac{1}{5} & \frac{1}{8} & 3 \\ 1 & 1 & 9 & 2 & \frac{1}{8} & \frac{1}{4} \\ \frac{1}{7} & \frac{1}{9} & 1 & 2 & \frac{1}{5} & 4 \\ 5 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{3} & 2 \\ 8 & 8 & 5 & 3 & 1 & 6 \\ \frac{1}{3} & 4 & \frac{1}{4} & \frac{1}{2} & \frac{1}{6} & 1 \end{bmatrix} \qquad (1)$$

A quick view of the matrix shows that row 5 clearly dominates the others, i.e. E seems to have the strongest motives, but what about the ranking of the other suspects relative to this criterion? Also, what about the other criteria of Opportunity and Profit? We take these two questions one at a time. We first rank the suspects relative to the motive criterion. This is where the idea of a dominant vector comes in to play.

If we apply the comparisons in matrix (1) consistently to any ranking of the suspects, then the effect of the comparisons should take effect and the rankings should tend towards the line determined by the dominant vector. (Try this. Use your CAS to multiply the equal ranking vector (1,1,1,1,1,1) by the 30-th power of the Motive matrix and normalize the result by dividing by the sum of the terms so that they add up to one). If you have down loaded the accompanying Derive© file called , Crime Solver, the program, `Dom_vec`, will find the dominant vector for any positive matrix up to a given tolerance between the actual and computed value. Here is the result for the Motive matrix. Compare it with your result.

Dom_vec(Motive, 0.0001)=[0.12972, 0.14560, 0.08681, 0.12758, 0.41866, 0.09161]

Suspect E, corresponding to the fifth value in the list is clearly the highest ranked according to this criterion. Suspect B is a distant second followed by suspects A, D, F, and C.

We rank the suspects according to the remaining criteria. Once again we use the program, `Dom_vec`:

$$Opportunity = \begin{bmatrix} 1 & \frac{1}{4} & 3 & 8 & 5 & \frac{1}{8} \\ 4 & 1 & 9 & 9 & 6 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{9} & 1 & 5 & \frac{1}{7} & \frac{1}{6} \\ \frac{1}{8} & \frac{1}{9} & \frac{1}{5} & 1 & \frac{1}{8} & \frac{1}{9} \\ \frac{1}{5} & \frac{1}{6} & 7 & 8 & 1 & \frac{1}{6} \\ 8 & 2 & 6 & 9 & 6 & 1 \end{bmatrix}$$

With Dominant Vector:

Dom_vec(Opportunity, 0.0001)=[0.13351, 0.28720, 0.03993, 0.01829, 0.09050, 0.43054]

We move on to the Profit motive.

$$Profit = \begin{bmatrix} 1 & \frac{1}{6} & \frac{1}{4} & \frac{1}{9} & \frac{1}{7} & \frac{1}{4} \\ 6 & 1 & \frac{1}{7} & \frac{1}{9} & 1 & \frac{1}{3} \\ 4 & 7 & 1 & \frac{1}{3} & \frac{1}{5} & 4 \\ 9 & 9 & 3 & 1 & 7 & 8 \\ 7 & 1 & 5 & \frac{1}{7} & 1 & 6 \\ 4 & 3 & \frac{1}{4} & \frac{1}{8} & \frac{1}{6} & 1 \end{bmatrix}$$

With Dominant Vector:

Dom_vec(Profit, 0.0001)=[0.02237, 0.06672, 0.15117, 0.48649, 0.21275, 0.06048]

Upon examining the dominant vectors for the three rankings, you see that each criterion gives a different ranking of the three suspects. The team of detectives has one more "collective decision" to make: How does the evidence fit the crime they are investigating? In other words, how much weight should they give to each of the three criteria relative to the circumstances of this crime? Once again, they can use the Analytical Hierarchical Ranking Scheme to assign the weights.

The matrix arrived at by the detectives is:

$$Priorities = \begin{bmatrix} 1 & 5 & 8 \\ \frac{1}{5} & 1 & 4 \\ \frac{1}{8} & \frac{1}{4} & 1 \end{bmatrix}$$

With Dominant Vector:

Dom_vec(Priorities, 0.0001) = [0.73340, 0.1991, 0.06754]

It is clear that Motive and Opportunity are the dominant criteria in this case and the Profit has relatively little role to play in the opinion of the detectives. This should be expected in the murder of a child coming from a middle class family. Indeed, Motive is by far the prime criterion. Each individuals score for the three criteria is multiplied by the weighting given to the criteria to find a final score for each suspect. This can be done using the Derive© program `MOP_Suspects` that is a part of the package supplied with this activity. The result of the program is given below.

MOP_Suspects(Priorities, Motive, Opportunity, Profit, 0.00001)

$$\begin{bmatrix} 0.33943 & E \\ 0.16847 & B \\ 0.16599 & F \\ 0.13007 & D \\ 0.12322 & A \\ 0.01812 & C \end{bmatrix}$$

Upon further investigation and questioning, it turned out that suspect E had committed the murder and confessed to the murder. He was arrested, tried, and found guilty.

Exercises Related to the Activity

1. Read a mystery novel and prior to reaching the end of the novel, identify the major suspects. Use the Analytical Hierarchical Ranking Method to identify the possible perpetrator of the crime. Compare your result with the author's solution of the crime.

2. Together with a small group of friends, watch a detective show on television that has a good development of the evidence. Have everyone leave before the end of the show, but videotape the ending portion that you missed. You and your friends create the matrices for the Analytical Hierarchical Ranking Method and determine the most likely guilty suspect. Watch the end of the show and compare your result.

3. The Analytical Hierarchical Ranking Method has applications in many areas. Apply the method to making an important decision in your life, such as choosing a college to attend, choosing a summer job, choosing a destination for a family trip, etc. Obviously you will use criteria other than the ones used for crime solving.

4. Choose important criteria for success in a particular sport. Rank the teams in your league relative to these criteria using the Analytical Hierarchical Ranking Method. How do the results compare to the final rankings of the team after the end of a season?

Final Comments

The role of the coroner ended in this case at the point the coroner assigned an manner (murder) and means (gun shot wound to the chest) to the death of the victim. The remainder of the work was for the police. They did the interviews and gathering of evidence.

Crime solving, like the practice of medicine, is not a science. Scientific methods are applied to the evidence and conclusions are reached. However, it is seldom that the

results are based on scientific laws or indisputable facts. There is a lot that requires the insight and training of the investigator. The methods used in this activity are a way for the investigators to organize their insights and "hunches". They are nothing more. They can be used to direct an investigation, not produce irrefutable evidence. In the case of the above investigation, this organization produced a focus that lead to a confession by the guilty party.

Bones of Contention

Using Skeletal Remains as Clues to Identity

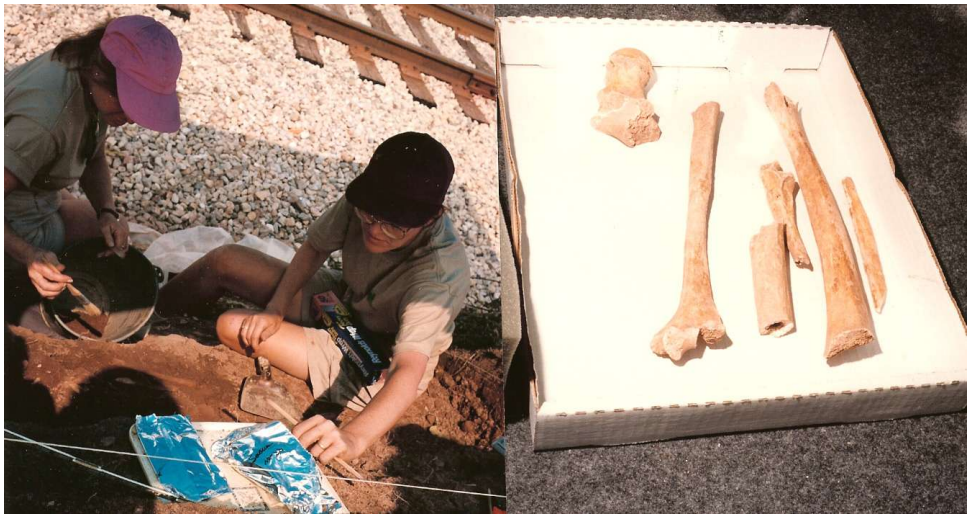


Photo On Left: National Park Forensic Anthropologists digging at the site of the discovery
Photo on Right: The bones recovered March 19 and 20, 1996 along the railroad cut on the Gettysburg Battlefield

SCENARIO (First Person Account by the Coroner)

In March of 1996 a visitor to the Gettysburg National Park from the state of Oregon was hiking along a part of the northern edge of the Civil War Battlefield known as the "railroad cut", the scene of intense fighting on the morning of the first day of the battle, July 1, 1863. He spotted bone fragments on the slope of the railroad cut. The National Park Service Representatives at Gettysburg were notified. The bone fragments were taken to the Smithsonian Institute in Washington, DC to be examined by a forensic anthropologist. The anthropologist's report was received in August of 1996 and appropriate officials of Adams County, the County where the battlefield is located, were notified. Since I was Coroner of Adams County, I was notified. We were taken to the site of the discovery. The report from the forensic anthropologist at the Smithsonian Institut concluded that the bones were those of a male, aged 20 – 25 years of age, who was killed by a gunshot wound to the head delivered by a rifle commonly used during the Civil War era. The bones were over 100 years old and most likely belonged to a soldier in the Battle of Gettysburg.

By mutual agreement the jurisdiction of the investigation was released to the National Park Service and a decision was made to initiate an archeological dig of the site. The project was conducted during September, 1996 by the National Park Service, the Denver, Colorado Service Center Applied Archeology Center, the Gettysburg National Military Park, and GAI Consultants of Monroeville, Pennsylvania.

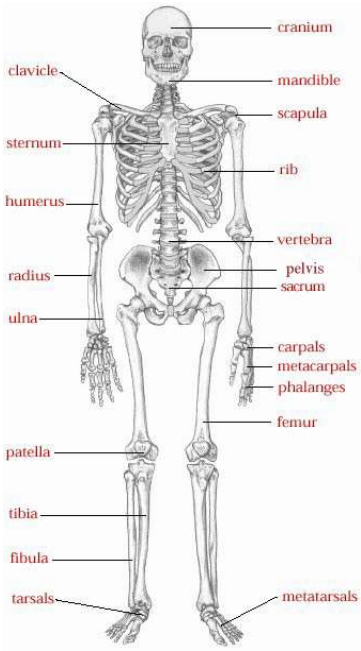
BACKGROUND

Sometimes when the coroner is called to a scene, especially one related to a crime that has occurred a long time ago or to a location that has been a 'dumping ground' for bodies, all that remains is a skeleton, or worse yet, a partial skeleton. It may seem that the identification of the person whose remains have been located is an impossible task. However, with the aid of a good forensic anthropologist, it is possible to determine the approximate age, the sex, the ancestry, and approximate stature of the individual. This information, in and of itself, is not enough to completely identify the individual, but it can give pointers that can lead to the identification. Much of this information can be determined even if the investigator only has access to a partial skeleton, such as the one shown in the picture above.

In the following material you will learn how bones can tell an investigator about the person they came from. If you are fortunate enough to have a living skeleton in your school, that is one that is made or cast from bones of a person who had been alive, you may be able to discover some characteristics of the person at the time of their death using the material in this activity. Even if you do not have a living skeleton available, you will perform some, but not all, of the measurements and observations on your live class mates and check your results against the mathematical tools that forensic anthropologists and pathologists use.

Our first job is to learn a little about bone anatomy. Below are two views of a skeleton, an anterior (front) and a posterior (back) view. The bones are labeled. Look at their names and position in the skeleton. These diagrams were taken from the web site for the Hillendale Elementary School Health Program in Chadds Ford, Pennsylvania at the URL.

http://hes.ucf.k12.pa.us/gclaypo/skelweb/skel04.html



The most important bones for determining the stature of the individual are: the femur, the humerus, the tibia, and the fibula. You will be using the first three of these in your activity.

For determining sex and ancestry, the skull, the pelvis, and the sacrum play an important role.

The degree of ossification of the cartilage at the head of the bones can help in determining the age of the person at their time of death.

Determining the Age at Time of Death

This is, perhaps, the hardest of the traits to determine with any accuracy. It is relatively easy for an experienced observer to make a good estimate if the person has died prior to reaching maturity, Translated to numerical values, this signifies ages from birth until the early twenties. The reason for this is that bones are growing during this time. Between the shaft of the long bones and the head are growth centers, i.e. areas stimulating a bio-chemical reaction that produced bone growth. These ends are covered by layers of cartilage associated with areas called growth plates or *epiphysis*. These are located at the end of the shaft of the long bones. The cartilage provides materials for the growth center to work on and turn into bone. Although this process occurs during the entire time period from birth to maturity, it is most active during the ages of 17 until 20. Thus for individuals who have not reached maturity, the amount of cartilage found at the ends of the long bones can give a reasonable approximation of the age at death.

After a person reaches maturity, the determination of age becomes more difficult. Investigators can determine the amount of deterioration that has taken place in an individual's bones, but that is not an accurate measure. It is affected by an individual's nutritional habits, weight, exercise, and general health as well as other factors. The best that can be done is to determine a mean age for all individuals who have exhibited an amount of bone deterioration similar to that of the skeleton.

The topic of more accurately determining the age of the deceased from skeletal remains is an area of forensic anthropology where great activity is taking place. Other areas of the skeleton are being examined, such as the iliac crest and the ileum of the pelvis. DNA analysis of bone tissue is making some progress. One approach is the investigation of the length of sequences of amino acids composed of the three nucleotides GAG in certain regions of DNA from bone tissue. These sequences seem to exhibit a parabolic growth pattern with a vertex corresponding to ages in the mid forties. At the present time, the results only yield a mathematical average age estimate.

A skeleton's teeth can also give evidence of an approximate age. Since different teeth erupt at certain ages in a young person, the completeness of a persons teeth can give a fairly accurate estimate of age. For a skeleton with a full set of teeth observation of wear on the teeth and also missing teeth may give some indication of the age at the time of death.

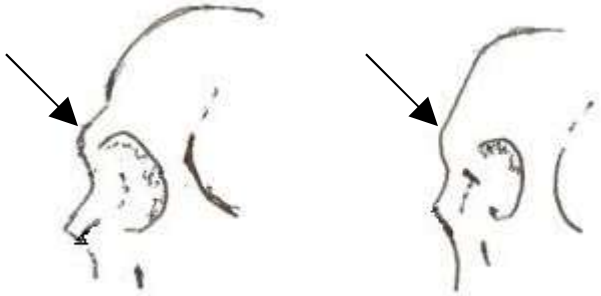
FINAL NOTE: If you have access a living skeleton during this activity, look at the ends of the femur, tibia, and humerus for evidence cartiledge. Also look at the teeth of the skeleton. It is highly unlikely that you will find anything interesting since most living skeletons are obtained from mature individuals. Also, the suppliers of these teaching devices generally replace any missing teeth.

Determining the Sex of the Deceased

For most primates the determination of the person's sex is a binary process. The individual specimen was either a female or not. The answer is either YES or NO, TRUE or FALSE, etc. One of the advantages of this situation is that it is possible to use qualitative, as opposed to quantitative, observations to determine the sex of a skeleton. The areas of special interest are the skull and the pelvis. We will also use the femur to help identify sex when the skull and pelvis may not be available to the investigator.

We begin with the skull. There are three areas of interest, the eyebrow ridge, the mandible, and the posterior base of the cranium, the occipital area, that protects the cerebellum during life.

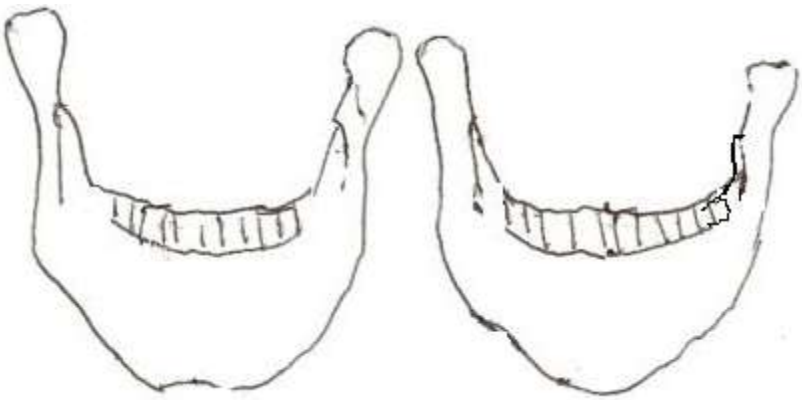
We begin by looking at sketches of two lateral (side) views of the skull, one of a male and one of a female.



Male **Female**

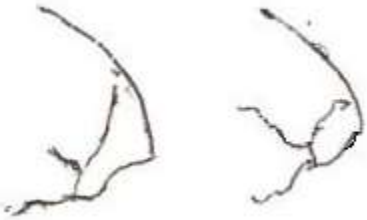
Although slightly exaggerated in this sketch, it is the case that the eyebrow ridge of a male is more prominent than that of the female. Thus, a reasonably flat area above the eye socket would support a hypothesis that the skeleton belonged to a female. A prominent bump would indicate a male skeleton.

Next we look at the mandible. The base, or chin area, of a female is more smooth and rounded than that of a male. The base of the male mandible tends to be more angular.



Male **Female**

Finally, the occipital profile of the skull reveals another difference between males and females. Once again the female profile appears to be smoother than that of the male.



Male **Female**

The most telling and, hence, easiest to read portion of the skeleton with regard to the sex of its owner is the pelvis. At the onset of puberty, the female pelvis begins to change shape. This is in reaction to the presence of newly secreted hormones, and is to allow females to give birth by expanding the pelvic outlet. The male skeleton also reacts to hormonal changes at puberty, but this is largely in parallel with increased muscle mass and the subsequent effect of increased forces on the skeleton.

Female Pelvis

Anterior View

Posterior View

Male Pelvis

Anterior View

Posterior View

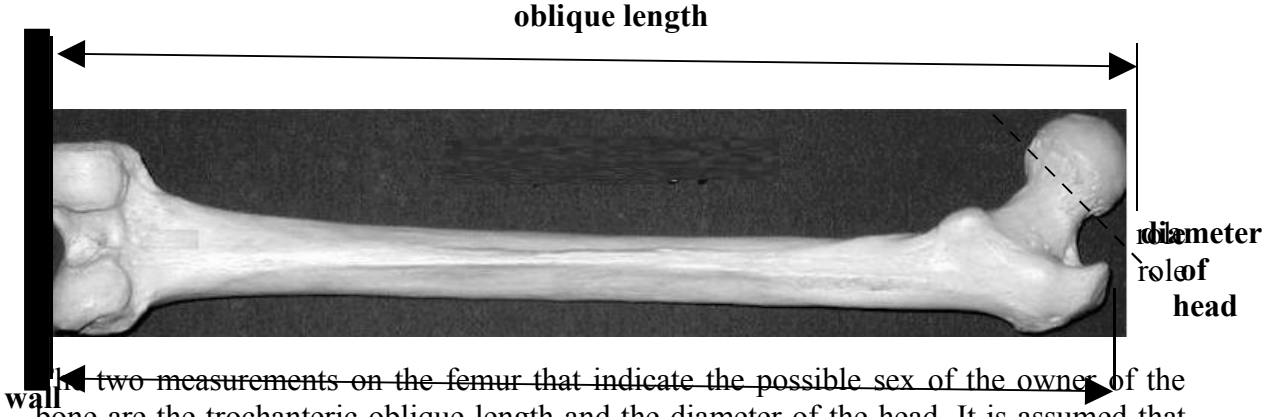
These views were adapted from the BBC site at URL
http://www.bbc.co.uk/science/humanbody/body/factfiles/pelvis/female_pelvis.shtml

In the illustrations shown above, it is immediately obvious that the male pelvis appears to be larger than the female pelvis. The female pelvis is also proportionally shallower and wider than the male's pelvis. There are other, more reliable indicators. These are the pubic arch and the sacrum. Both of these change as the female matures to accommodate the birth canal.

Looking at the anterior views of the two pelvises, it is apparent that the pubic arch makes a wider angle on the female pelvis than on the male pelvis. A general rule of thumb is that if the angle is greater than 90°, the skeleton belonged to a female; less than 90°, it belonged to a male. Some anatomy books place the demarcation at 70°.

The other indicator for the sex of the owner of the skeleton is shown clearly in the posterior view of the pelvis. It is the sacrum. The female's sacrum is wider, shorter, and less curved than the male's sacrum. This fact will be quantified in the next section on determining race using a ratio, called the *sacral index*.

Our final indicator of the sex of the owner of the skeleton is the femur. The head of the femur is placed against a stationary board and the measurements indicated below are carefully made.



The two measurements on the femur that indicate the possible sex of the owner of the bone are the trochanteric oblique length and the diameter of the head. It is assumed that we are dealing with a bone from a mature individual, i.e. no epiphysis.

Sex	Trochanteric Oblique Length	Diameter of Head
Female	< 39 cm	<4.15cm
Male	>45 cm	> 4.55 cm
Indeterminate	40.5 to 43 cm	4.35 to 4.45 cm

Note that there are length intervals that are not covered by the table. In these cases conclude "probably female," or "probably male" based on the proximity to the ranges for the sex that is closest to the given measurements. If the measurements are in the Indeterminate range, the determination of sex can not be made with any accuracy.

Determining the Ancestry of the Deceased

Once again we will start with the skull. This is the most accurate indicator of race. We will use three classifications for race: European descent, African descent, and Mongoloid descent. Mongoloid includes Asians, American Indians, and those of non European or African descent.

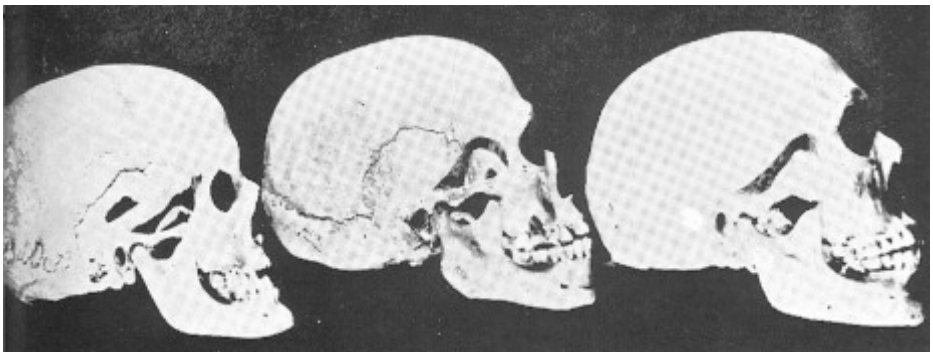
The following table details some of the differing characteristics of skulls from individuals of these three descents. Only some of the more easily identified characteristics are given here.

Differing Skull Features That Indicate Ancestry

Feature	European	African	Mongoloid
Profile	Straight	Extended jaw	Intermediate
Height	High	Low	Middle
Width	Narrow	Narrow	Broad
Face Width	Narrow	Narrow	Very wide
Face Height	High	Low	High
Nasal Opening	Narrow High	Wide Low	Narrow High
Palate Shape	Narrow Wedge	Wide U-shape	Moderate Wide

Like sex, the characteristics relative to the ancestry of a skull are discrete. Furthermore, many are judgment calls. So it is best to use a table using the characteristics and categories above to mark either with a check or a blank. Then you can make an "informed decision."

Below is a picture of three skulls, one from each ancestry category. Can you correctly identify which category is represented by each skull?



Another bone of importance is the sacrum. In fact it is not a bone, but three fused vertebrae. It derives its name from the ancients who believed that the body made a sacred journey to the land of the gods. Because of the thickness and size of this bone, it was one of the last to decay. Thus they believed that the bone was sacred and, hence, was the building block for the new body of the deceased.

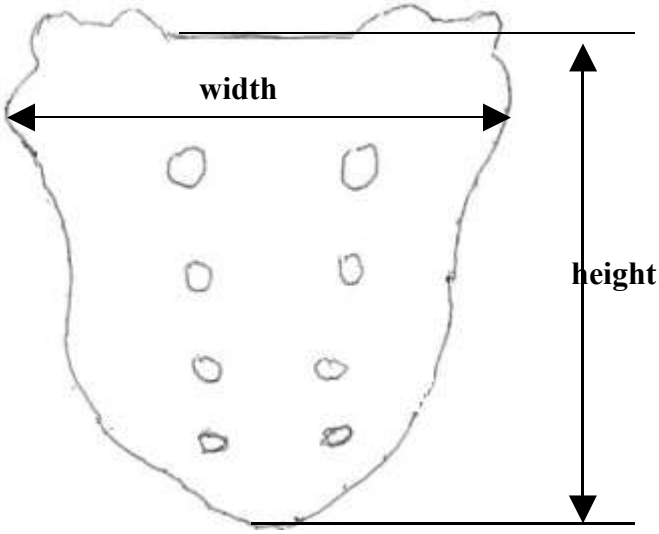
Looking at the sacrum there are two measurements that are used, it's width and it's anterior length. Since the sacrum curves away from the main line of the spine finding its anterior length may be difficult. It is suggested that one use a metric tape measure or a string that can be carefully marked. The sacral index is computed by the following formula.

$$sacral\ index = \frac{sacral\ width}{sacral\ anterior\ length} \times 100$$

.

The rough sketch of the anterior sacrum shown below has these two measurements marked.

Sacrum Anterior View



The following table gives a listing of some average values for the sacral index.

Average Values for the Sacral Index

Ancestry	Female	Male
African	103.6	91.4
European	112.4	102.9
Japanese	107.1	101.5

This table is limited in many respects since it lists only three ancestry groups, one severely more limited than the others. It also lists only the average value and gives no standard deviations to attach a probability to an observation. On the other hand it may give good evidence of whether the deceased was of African or European descent.

Another indicator of ancestry is the femur. One indicator is the anterior to posterior curve of the bone. Another is the torsion or twisting of the bone between the head of the femur and where it joins the knee.

Characteristics of the Femur that Indicate Ancestry

Characteristic	African	European	Mongoloid
A/P curvature	Slight	Bowed Forward	Curved outward
Torsion	Untwisted	Twisted	Very Twisted

Determining Stature of the Deceased

There are many methods for estimating the height of the deceased using the length of different long bones or combinations of the lengths of long bones. In this section we will discuss only one of these methods. That is one using the oblique length of the femur. The reason for this is that the length of the femur is the measurement that correlates best with the living height of the individual.

The formulae used by forensic anthropologists depend on the ancestry of the individual. These formulae are given below. All measurements and estimates are in cm.

African: $2.10 \times$ (oblique length of femur) $+72.22$
European: $2.32 \times$ (oblique length of femur) $+ 65.53$
Mongoloid: $2.15 \times$ (oblique length of femur) $+ 72.57$

These estimates are generally accurate to within $\pm$ 3 cm.

As mentioned earlier, there are estimates using other long bones. The most commonly used involve the humerus, the tibia, and the combined femur and tibia.

Forensic Objectives

- To understand how careful examination can assist in the discovery of certain essential facts about a deceased individual when only skeletal evidence exists. These facts can lead to the identification of the deceased..
- To organize a careful forensic evaluation of skeletal evidence.

Scientific – Mathematical Objectives

- To organize a careful investigation of scientific evidence.
- To carefully collect data and analyze it using statistical tools that are available in your CAS.
- To test the results of established forensic anthropology formulas.
- To use regression analysis to construct a formula for determining stature of an individual based on the length of the individual's femur.

Materials

- Derive version 6 or another system that can compute linear regression equations from paired data.
- Preferably a measuring tape with metric markings. If such a tape is not available, a meter stick and a string can be used.

Activity 1 – If a living skeleton is available

1. Using a spread sheet or word processing program, make up a template that contains all of the features discussed in the background section for determining the sex, ancestry, and stature of your living skeleton. Organize the template in a way that makes anatomical sense. Have your instructor approve your template before proceeding.

2. Using your template, record your observations and measurements.

3. If any of your categories require a calculation perform the calculation.

4. On your template include a column for a conclusion based on your observations and calculations.

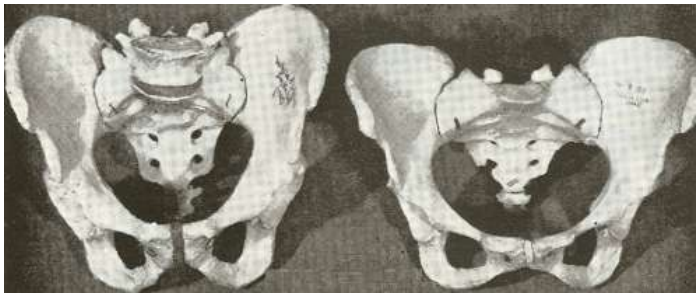
5. Report your conclusions to your instructor.

Activity 2 – Questions on Reading and a Laboratory Activity

1. Using the picture of the femur given in the section on determining the sex of the deceased, make a judgment on the ancestry of the deceased. Why did you make this choice?

2. In the section on estimating the stature of the deceased it was specified to make the measurements in cm. However, in the section on determining the sacral index, no units were specified. Why?

3. In the picture shown below, identify each of the pelvises with respect to their sex. Justify each of your decisions.



4. In the picture of the tree skulls shown in the section on determining ancestry, one of the skulls is obviously one of a male. Which one is it? Explain.

Laboratory Activity

1. Work with a partner. Ask your partner to stand erect, lock their knee and swing the leg on the side that you are facing. The pivot point should give you an accurate estimate of the location of the head of the femur for that leg.

2. Have your partner bend their knee. Identify the knee cap. Locate the middle of the knee cap.

3. Carefully measure the length of the distance from the head of the femur to an imaginary line parallel to the floor and passing through the middle of the knee cap in cm. This should give a reasonable approximation to the length of the femur. Record this distance.

4. Ask your partner to stand erect and measure your partner's height in cm. Record this value.

5. Use the appropriate formula from the section on computing the living stature from the length of the femur to compute an estimate to your partner's height. Record this value.

6. Compare the estimate of your partner's height to the actual height of your partner. Is the estimate within the predicted 3cm of the actual height? If not, give a reason or reasons why this may be the case.

7. Switch places with your partner and repeat steps 1 – 6.

8. When all pairs have completed their measurements and calculations, collect all of the class data concerning the femur length and actual height of each member of the class. Note the Ancestry group for each data point. Divide the data into ancestry groups.

9. Divide the data into ancestry groups. At a minimum there must be two sets of data in each ancestry group to do a linear regression. Why? .

10. For each ancestry group perform a linear regression (FIT) wbith femur length as the independent variable and height as the independent variable. Compare the result with the formula from above that is given for the group.

11. Using different colors graph the regression line and the line used by forensic anthropologists. Draw a band representing the line ± 3 cm about the forensic anthropologists line. Use dotted lines to represent this band.

12. Does your regression line fit within this band in the range of femur lengths measured by you and your classmates? What does this tell you?

CORONER'S COMMENTS

I have been involved in other cases where only skeletal remains were found, but this particular incident received a great deal of attention because of its historical interest. On January 2, 1998 the Hanover Evening Sun newspaper in Hanover, Pennsylvania ran an article based on the Smithsonian Report and also the investigations of a John W. Busy, an amateur historian from Virginia who specialized in researching the rosters of Civil War military units. A sidebar to the story contained the following information about how the Smithsonian findings were made: The story and sidebar were written by T. W. Burger, an Evening Sun Reporter.

- Density of bone, lack of arthritic changes, and fully developed molars indicated an age of 20 – 25.
- Muscle development of the right and left arm was consistent with right-handedness [Note: muscle development can be determined by the size and thickness of the arm bones.]
- He was most likely a farmer because he was strong and healthy, and because that is what most Americans did for a living in the 19-th century.
- He also rode horses frequently, as indicated by friction marks on the joints of the legs.
- As a baby, he had suffered chicken pox , colds and probably the mumps, judging by marks on his teeth.
- The man's diet was good, based on the fact there was no evidence of iron deficiency in the bones. His teeth showed no signs of cavities, indicating no sugar to speak of in his diet. That, in turn, indicated that he was not from the wealthier classes who could afford to buy sugar and were thus prone to tooth decay.

- Smithsonian researchers found skull fracture particles and x-ray evidence of lead particles consistent with head trauma in the form of a gunshot wound to the head.

The main article stated that Mr. Busey's research showed that the 2-nd and 42-nd Mississippi fought at the railroad cut. The 42-nd sustained especially heavy losses. There were 5 or 6 farmers in that unit. Mr Busey said the physical characteristics of the owner of the bones lead him to believe the soldier was Jim Bonds, aged 24, a farmer from Jacinto, Mississippi. There is no positive proof of this identification.

When Will They Ever Change?

An Elementary Introduction to Fingerprint Analysis

Scenario

You have been called to the scene of a one car motor vehicle accident. It had begun to rain just a few minutes before the accident. This is the time when the roads are most slippery. The car had skidded off of the road and into a utility pole. The only apparent victim is the driver of the vehicle who exhibited multiple trauma injuries as a result of being thrown through the front windshield of the car. His face is severely lacerated making photo identification highly improbable. No identification can be found on his person or in the vehicle. There is evidence that others had been riding in the car, but they had left the scene of the accident and could not be located. After examining the body, you pronounce the driver to be deceased on the scene of the accident and begin your investigation. Part of your responsibility is to identify the deceased and notify his family. Since there is no apparent way to quickly identify the victim, you decide to have fingerprints of the body taken in the morgue and send them for possible matches in the fingerprint database. It turns out that when the victim applied for a temporary worker permit before crossing the border, he was fingerprinted and the fingerprints were added to the database. As a result a positive identification was made, and you received information on the victim's identity, the name of his host employer, and a contact for his family in his home country.

Background

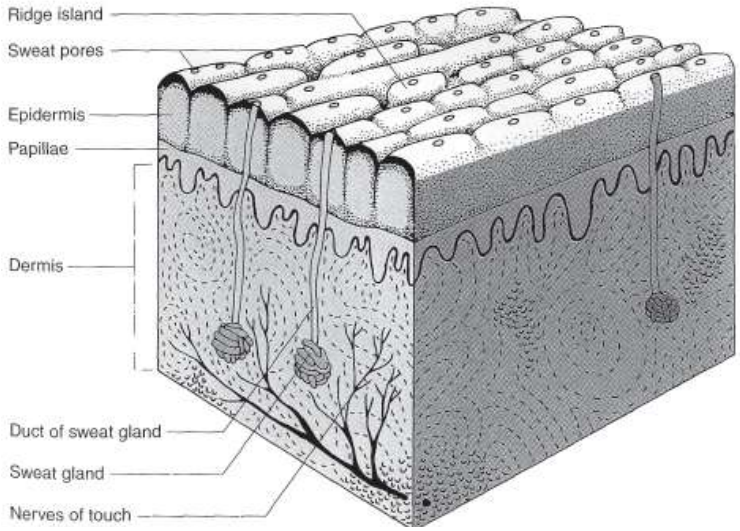
Fingerprints were used for identification purposes in the United States in 1901 when the New York City Civil Service Commission required that all of its applicants be fingerprinted for identification purposes and the prints classified and stored in a central location. This was four years after the process of fingerprinting was adopted in England by Scotland Yard as a means of identifying criminals and used in detective investigations. This followed years of lobbying by Sir Francis Galton, half-cousin of Charles Darwin, and a polymath, anthropologist, eugenicist, tropical explorer, geographer, inventor, meteorologist, proto-geneticist, psychometrician, and statistician. Prior to 1896, a system of making detailed measurements of certain physical characteristics of an individual was used for identification purposes. This system proved to be flawed when two prisoners at the federal prison in Fort Leavenworth, Kansas who were unrelated, but had similar names and appearances were shown to have the same physical measurements, but different fingerprints.

Fingerprints are actually friction ridges that develop on the skin of the hands and feet of human beings. They are with us from birth and do not change as we age. Some people have tried to change their fingerprints by rather painful processes such as slashing them with a knife, or as in the case of the 1930's American gangster, John Dillinger, burning his finger tips with acid. The pattern of the ridges still remains. In Dillinger's case the fingerprints taken at the morgue after he was murdered agreed with ones from a much earlier time taken when he had been arrested.

The following diagram shows how fingerprints are formed. The fingerprints develop at the time the dermal papillae develop in the human fetus. NOTE: Even though a newborn's fingerprints are developed, hospitals use babies' foot prints for identification instead of fingerprints. It is simply much easier to take the foot print of an infant.

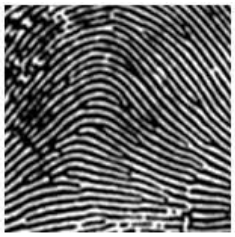
This diagram is taken from the New Mexico Department of Health website and available at the URL.

http://dhi.health.state.nm.us/elibrary/cchspmanual/fingerprint_manual.pdf



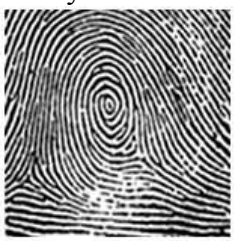
The important fact is that even though we may cut our fingers or damage them in other ways, the underlying ridge patterns never change.

The first step in fingerprints is to recognize the common patterns associated with the ridges. The following diagrams are basic and the patterns are very clear.



Arch

Loop



Whorl

Loops make up about 60% of all fingerprints. Whorls make up about 35%. The remaining 5% are arches. But life is not simple at even this level. There are two kinds of Arches: the one shown above and a Tented Arch. There are two kinds of Loops: an ulnar loop that points towards the little finger and a radial loop that points towards the thumb. Sometimes these are called Right and Left Loops. The above picture is of what may be called a Left Loop. The Whorls are the most diverse. There are four kinds of whorls: the plain whorl (shown above), a pocket loop (looks like a loop, but has a small whorl in the center), a double loop (a radial and ulnar loop intertwined), and an accidental whorl (a combination of two or more patterns or any other unclassified pattern). One percent of all fingerprints are accidental whorls.

Tented Arch Right Loop Double Loop Pocket Loop Accidental Whorl

The first two images on the left as well as the three shown previously were found in a thesis by Sharat S. Chikkerur at the URL:

http://web.mit.edu/sharat/www/research/thesis.pdf

The three images on the right are from the New Mexico Department of Health website cited above.

Thus we have eight general classifications of fingerprints. Sir Edward Richard Henry in 1897 proposed a system for classifying records of individuals' ten fingerprints so that a search of the fingerprint records, or fingerprint database, could be narrowed down. The United State Federal Bureau of Investigation (FBI) refined Henry's system and this method is generally used by most police agencies today.

The full FBI system is somewhat complicated, but it begins with a primary classification system that is rather easy to explain. This primary classification will serve the purposes of this activity.

The system matches up the ten fingers of the average human and creates the following "fractions."

$$\frac{\text{R. Index}}{\text{R. Thumb}} \quad \frac{\text{R. Ring}}{\text{R. Middle}} \quad \frac{\text{L. Thumb}}{\text{R. Little}} \quad \frac{\text{L. Middle}}{\text{L. Index}} \quad \frac{\text{L. Little}}{\text{L. Ring}}$$

The primary classification is based on the presence of a whorl in each of the fractions. If the first fraction has a whorl in either the numerator or denominator, the corresponding part of the fraction is given value of 16. If it has a whorl in either part of the second, it is given an 8; in the third, a 4; in the fourth a 2; and in the fifth a 1.

The values of all of the numerators are all added up and a value of 1 is added to the total. The same holds true for the denominators. Thus, if a whorl is found on the Right Ring

Finger and the Left Index Finger. The primary classification value for the set of fingerprints is given a value of:

$$\frac{0+8+0+0+0+1}{0+0+0+2+0+1}=\frac{9}{3}$$

The fraction is not reduced.

Roughly 25% of the results fall into the $\frac{1}{1}$ category, i.e. all ten fingerprints are either loops or arches.

In addition to the primary classification scheme, there are other considerations of the print. These have to do with the individual ridges themselves. These characteristics are called *minutia.* They have to do with breaks in the continuity of the ridges and branching of a ridge line.

The illustrations below show the eleven most common primary classifications of minutia.

Ridge dot:

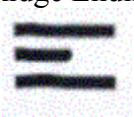
Bifurcation:

Double Bifurcation:

Opposed Bifurcation:

Trifurcation:

Ridge Ending:



Ridge Crossing:

Lakes:

Islands:

Hooks:

Bridges:

These characteristics were defined and illustrated at the URL
www.virtualsciencefair.org/2004/fren4j0/public_html/ridge_characteristics.htm

If each fingerprint of an individual is divided into four quadrants and a score is kept on the number of the above minutia patterns shown above we can see that when the number of possibilities for these numbers and the various primary classification possibilities, we can see that there is a very large number of possibilities for distinct fingerprint classifications of individuals.

FORENSIC OBJECTIVES

- To understand the nature of Fingerprints
- To identify the primary classifications of fingerprints
- To be able to use the FBI primary classification scheme for fingerprints
- To understand the minutia patterns of fingerprints

MATHEMATICAL – SCIENTIFIC OBJECTIVES

- To be able to compute the number of possible fingerprint patterns using the information given in the background section.
- To understand probability calculations and distributions
- To compare two probability distributions
- To see the value of having a CAS that can do unlimited integer calculations to do computations of large numbers

MATERIALS NEEDED

- A CAS set in *exact* calculation mode
- (Option 1) Tour your local police department and request that your fingerprints be taken as a part of the tour and that you receive a record of your impressions. Please check with your parents before doing this, your fingerprints will be registered in the fingerprint data bank.
- (Option 2)
  a. An ink pad such as one used for rubber stamps
  b. A file card similar in size to one that is used for recipes and divided into five columns and two rows (upper row for right hand digit impressions, lower row for the left hand).
  c. A method for cleaning your hands – soap and water with towels, cleansing towelettes, etc.
- (Option 3)
  a. A freshly sharpened No. 2 graphite pencil
  b. A pad of white paper (need not be too large 5cm X 5cm will do)
  c. A file card divided into five columns and two rows as described above.
  d. A method for cleaning your hands – soap and water with towels, cleansing towelettes, etc.
- A good magnifying glass
- A computer tool for recording data, tallying it, and doing appropriate statistical analysis.

Questions Related to Background Reading

1. Using the distribution of fingerprints given in the Background section, justify the statement that 25% of the FBI primary classifications are in the $\frac{1}{1}$ category?
2. How many different primary classification classes are in the FBI classification scheme? HINT: think binary.
3. If each fingerprint is divided into quadrants, assume that each minutia type can appear at most once in each quadrant. How many different minutia classifications can appear in each quadrant? In all four quadrants? On all ten fingers? HINT: Think in terms of the type appears in the quadrant or does not appear in the

quadrant. NOTE: This assumption is not really reasonable, but it makes the problem easier.

4. Combine the results of problems 2 and 3 to give the number of different fingerprints that are possible even in our simplified model. Compare this number to the population of the largest city in your country, your country, your continent, the world.
5. Are you convinced that each individual has a unique set of fingerprints and can be identified via their fingerprints? Discuss the reasons for your answer.

Activity

Prior to the Activity – Depending which of the above options you are using

Divide into groups of two students each. This is your investigative team. Be sure that each team has at least one magnifying glass.

Option 1

1. Get your fingerprint card from your teacher. Be careful not to smudge the prints.

Option 2 Each team needs two recipe cards and a stamp pad

1. Prepare your personal fingerprint card as described above. Put your name on the back of the card.
2. Test your stamp pad to make sure that ink will transfer to your finger tip. Wash your hand afterward.
3. Working with one person as the investigator (team member 1) and the other as the person being fingerprinted (team member 2. Team member 2 will extend their fingers in order. Team member 1 will roll each finger tip from the left side of the finger nail to the right side of the finger nail across the ink pad. It is important to ink the entire finger print.
4. Roll the finger tip in the appropriate row and column of team member 2's fingerprint card. Wipe the excess ink from team membge er 2's finger tip.
5. Continue until all 10 of team member 2's digits have been fingerprinted.
6. Team member 2 will then wash her or his hands, removing all ink from the finger tips.
7. Switch roles and return to step 3.

Option 3 Each team needs two recipe cards, a thin pad of white paper, and a freshly sharpened Number 2 graphite pencil

1. Prepare your personal fingerprint card as described above. Put your name on the back of the card.
2. Turning the graphite pencil sideways heavily blacken an area on the top sheet of white paper large enough to roll a finger tip from side to side in that area. This will be your "stamp pad."
3. Test your stamp pad to make sure that ink will transfer to your finger tip. Wash your hand afterward.
4. Working with one person as the investigator (team member 1) and the other as the person being fingerprinted (team member 2. Team member 2 will extend their fingers in order. Team member 1 will roll each finger tip from the left side of the finger nail to the right side of the finger nail across the ink pad. It is important to ink the entire finger print.

5. Roll the finger tip in the appropriate row and column of team member 2's fingerprint card. Wipe the excess ink from team member 2's finger tip.
6. Continue until all 10 of team member 2's digits have been fingerprinted.
7. Team member 2 will then wash her or his hands, removing all ink from the finger tips.
8. Switch roles and return to step 4.

Classifying your fingerprints

1. Have your instructor approve your fingerprint card.
2. Using the magnifying glass, you and your partner examine each fingerprint on the cards and classify it as an arch, a loop, or a whorl. Do not write the classification on the card.
3. Enter the finger identification and the fingerprint classification in a spreadsheet or text document on your computer or hand held device.
4. Determine the FBI Primary Classification for your fingerprints.
5. Transfer your tally to your instructor or class record keeper, who will construct a class tally of all different fingerprint types. Create a pie chart showing the distribution of the class' fingerprints.
6. How does your class distribution compare to the global distribution given in the Background?
7. Ask your teacher to photo copy all of the fingerprint cards and give a copy of the photocopied cards to each investigative team member of the class
8. How many different FBI Primary Classifications were in your collection of class prints? How many were in the $\frac{1}{1}$ category?
9. You and your partner will create files to store the cards for each category of fingerprints found in your class.

Identify the "culprit"

1. Your instructor tells you that someone had been in the classroom last evening and left a set of fingerprints on the chalk board. The teacher has previously picked out a card from the class fingerprint cards and describes the pattern on each finger.
2. Compute the FBI Primary Classification from the teacher's information.
3. Using the file corresponding to the FBI Primary Classification, determine if there is more than one possible suspect in the class.
4. If it is the case that you have more than one suspect. Note the differences between the fingerprints on the cards in that class. Try to pick out unique characteristics for each set of fingerprints belonging to the suspects in the class.
5. Create a list of no more than 10 questions related to the differences you have spotted for your teacher to answer for you.
6. Based on these answers, see if you can identify the culprit.

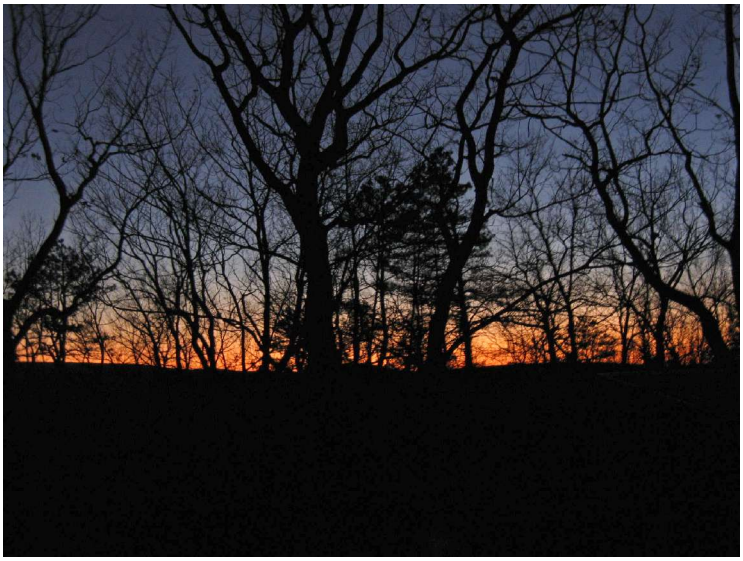
Coroner's Comments

Generally, it is not necessary to take the fingerprints of the deceased. Many times there are witnesses to the incident who may know the deceased. This is a starting point for the identification process. The deceased may be carrying some means of identification. It may be able to trace the vehicle back to a dealer via its registration. The vehicle identified by family, co-workers, or friends. If the deceased has any unusual physical characteristics or a tattoo, that can be used to start the identification process. However, fingerprints, if available and if they are in a fingerprint database, provide a positive identification process.

All individuals entering the military, police, or civil service professions are fingerprinted. Prisoners in state or federal prisons are fingerprinted. In the case of a death of any of these individuals the fingerprints of the corpse are compared to the previously taken fingerprints for absolute identification purposes.

Where In The World Is IT?

Understanding How GPS Uses Mathematics



SCENARIO

Two hikers have been on the trail for five days. Tomorrow they plan to reach their destination and return home from their wilderness vacation. It has been a wonderful week. The weather was great and their trail had a wonderful late fall appearance that is so different from that of the trail in summer. They make camp and cook their evening meal. On a nearby ridge the sunset is giving a spectacular display. The hikers walk over to see it. At this point their idyllic day ends. They discover some bones lying in the woods not far from their campsite. They appear to be human. The skeleton is incomplete. There are some leg bones and some ribs.

The terrified hikers decide that they must notify the emergency response organizations in the area. They have a pocket size GPS and a cell phone. They call 911 and report their finding and the GPS reading for the position of the bones. The emergency control center dispatcher consults the center's maps of the area and locates the campers' position. The police are notified. There is no need for medical personnel to respond to the scene. On the other hand you are notified, since there apparently is a deceased on the scene.

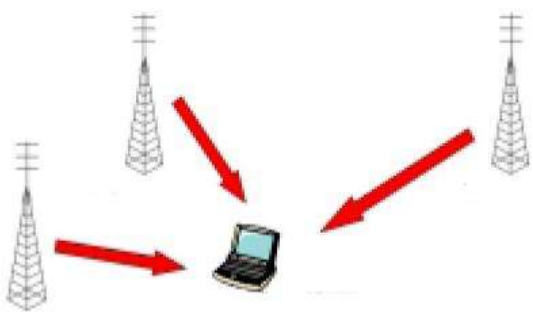
You call your deputy to accompany you to the scene, put on warm clothing and hiking boots, gather up a flashlight and your crime scene kit, and drive to the nearest access point to the hiking trail. After your deputy arrives, the two of you proceed to the scene. A brief examination indicates that the bones are not human. They appear to be those of a deer who died somewhere in the woods. At several locations on the bones teeth marks are apparent. Scavengers had feasted on the carcass and the bones were probably scattered as they were carried off for private dining.

BACKGROUND

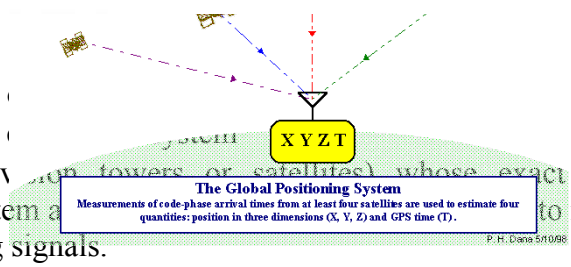
The use of a GPS in the above scenario was very important because it allowed the control center dispatchers to quickly locate the position on their maps and then alert and efficiently direct the personnel to the scene. This case was a bit of a wild goose chase, but if the emergency had been a real one, quick response could have made a huge difference. Even in this case it shortened a long, dark hike through the woods.

There are two general types of Global Positioning Systems (GPS) being used at present. The first is a 2–Dimensional system. This system uses the band width and time stamping of High Definition TV signals and can penetrate buildings and many areas that traditional GPS systems simply can not be used. They can only give longitude and latitude coordinates. They do not give the altitude of the location. As a consequence, they are primarily used for local applications where the altitude is relatively constant.

The other is a 3 satellites whose position is known. I e receiving unit. As a result this syst ose used by 2-D applications.



Both systems have s can penetrate buildings and obtain 3-D systems can give all three coord it have a "clear view of the sky."



The basic components

- A pre assigned
- Devices (Telev tower or satellites) whose exact positions within the coordinate system a to relay their positions by way of sending signals.
- A timing mechanism that is associated with the signal
- A unit that can receive signals from several devices and either quickly compute its distance from each of the devices or relay the information to a base station that can perform these calculations

Please note that we have not included any requirements for maps or said anything about navigating based on the calculations. This is a discussion of a *Positioning System*. Getting from here to there is another problem that also involves some interesting mathematics.

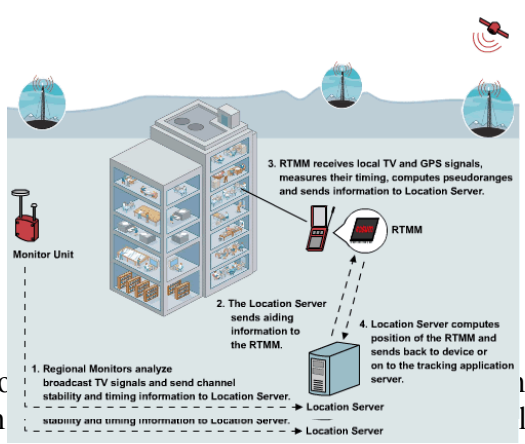
2-Dimensional GPS

In the United States and several other countries, 2-D GPS is used to keep track of the movements of prisoners who are placed on house arrest with strict limitations of their

movements. It is also used as a means for keeping track of children wearing GPS bracelets or anklets. It has even been responsible for helping to locate lost children and, quite probably saving lives.

Basically, GPS is nothing more than triangulation. A receiving device receives signals from three locations whose exact position is known and that have accurate clocks that are synchronized precisely with each other. The signaling devices are constantly sending their position and the time. The receiving device records this information and stamps it with the time that it received it. The time stamp does not need to be synchronized with the sending devices. From the information received from three sending locations and the time stamp the receiver can either calculate the x,y-coordinates of its position, or relay the information to a central processor that will do the calculations. The latter case is shown in the figure below.

The proposed modern systems rely on the synchronization signals that exist within the standard Digital Television transmissions. Thus, the information required for this type of GPS relies on already existing devices and does not require the construction of new transmitting towers.



Using Digital Televisio ntages over using satellite transmissions in certain ls have:

- On average about 40dB more power
- A wider band width
- Lower frequencies
- Superior geometry

This means that the signals can reach locations, such as inside buildings, that are inaccessible to standard GPS signals

Thus, this technology has applications that expand the available GPS applications. Here are two applications:

1. Locating the positions of first responders using a module located inside their emergency radios and portable signal generators attached to their belts.
2. Finding lost children in indoor locations, such as shopping malls or amusement parks, or in densely wooded locations using a bracelet or anklet device worn by the child.

Illustrating How The 2-Dimensional GPS Works

A child, wearing a "locator bracelet" has gone missing in a shopping mall. Transmitters located in the mall at positions 100 meters East and 200 meters North; 300 meters West and 100 meters North; and 200 meters East and 200 meters South of a central control station record the following distances from the child's bracelet:

Transmitter 1: 223.60680 meters
Transmitter 2: 200.00000 meters
Transmitter 3: 424.26407 meters

The following issustration shows how we can locate the approximate position of the "lost" child using Derive[tm] 6.1 graphics with slider bars.

The process is relatively simple. Since the digital tv signal waves travel outward from each transmitter in a uniform circle, we simply construct a coordinate system that contains the points (1, 2), the location of the first transmitter in hundreds of meters, (-3, 1), the location of the second transmitter, and (2, -2), the location of the third transmitter.

Then we enter the three equations of the circles leaving the radius as an unknown in each equation:

$$(x-1)^2 + (y-2)^2 = r1^2$$
$$(x+3)^2 + (y-2)^2 = r2^2$$
$$(x-2)^2 + (y+2)^2 = r3^2$$

As a penultimate step we move to the graphics screen and create a slider bar for each of r1,r2, and r3. Let the range of each slider bar go from 0 to a number large enough to include the distance from the child. Also, allow the increment to be 0.01 so we can slide the bar to the nearest foot of distance

Finally, move the first slider bar to 2.24, the second to 2.00, the third to 4.24.

While the distances are not absolutely accurate, the three circles appear to intersect at a unique point, (-1, -1). Thus, we locate the child at approximately 100 meters West and 100 meters North of the control station.

This solution is not necessarily an exact solution simply because we did not have enough pixels on the screen to give us better decimal accuracy, but the approximate solution got us close enough so that we can locate the child. Furthermore, the process of using slider bars gives us a feeling of how the GPS operates.

Solving The Equations Exactly

Radio and TV si et c denote the speed of light (c =

distanc ency)

or,

where δ = discrep

Let's consider the PS.

Transm

Lo

Obs

Transm

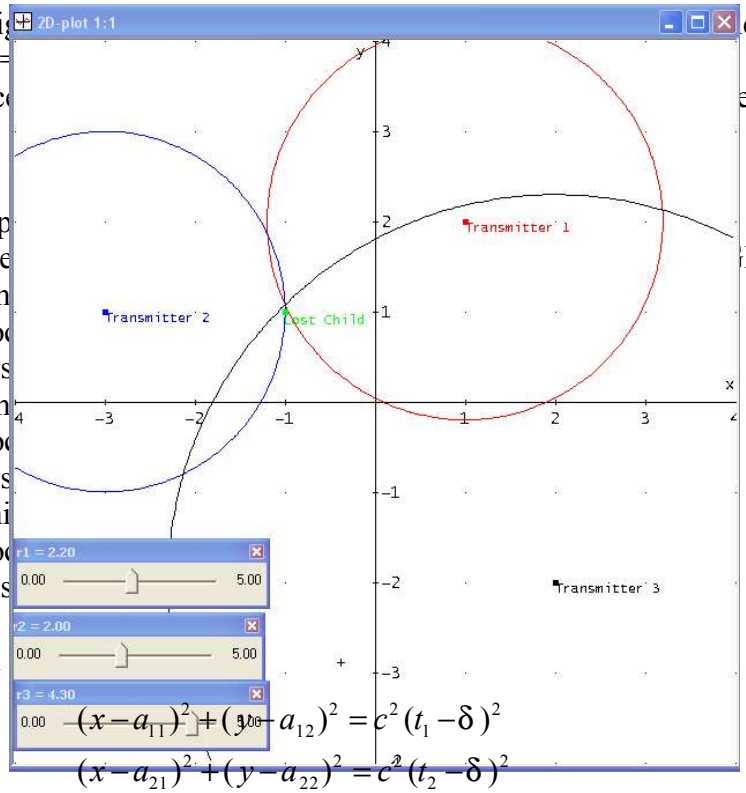
Lo

Obs

Transmi

Lo

Obs



This information

$$(x-a_{11})^2+(y-a_{12})^2=c^2(t_1-\delta)^2$$

$$(x-a_{21})^2+(y-a_{22})^2=c^2(t_2-\delta)^2$$

$$(x-a_{31})^2+(y-a_{32})^2=c^2(t_3-\delta)^2$$

NOTE: The geometric approximate solution is no longer available since we do not, at this stage, know the distance between the transmitter and receiver. All that we know is the time that the signal was received from the transmitter.

The system of equations that we have to solve is a system of three *non-linear* equations in three unknowns: x, y, and δ. The problem is that we only know how to solve systems of *linear* equations. One possible solution to our dilemma is to introduce a fourth transmitter and use the data and resulting equation from that transmitter to reduce the above system of non-linear equations to a system of three linear equations in three unknowns.

The problem with this solution is that a fourth transmitter may not always be available and, even if one is, there is the problem of synchronization of this transmitter with the existing three transmitters, i.e. we are increasing the probability of error.

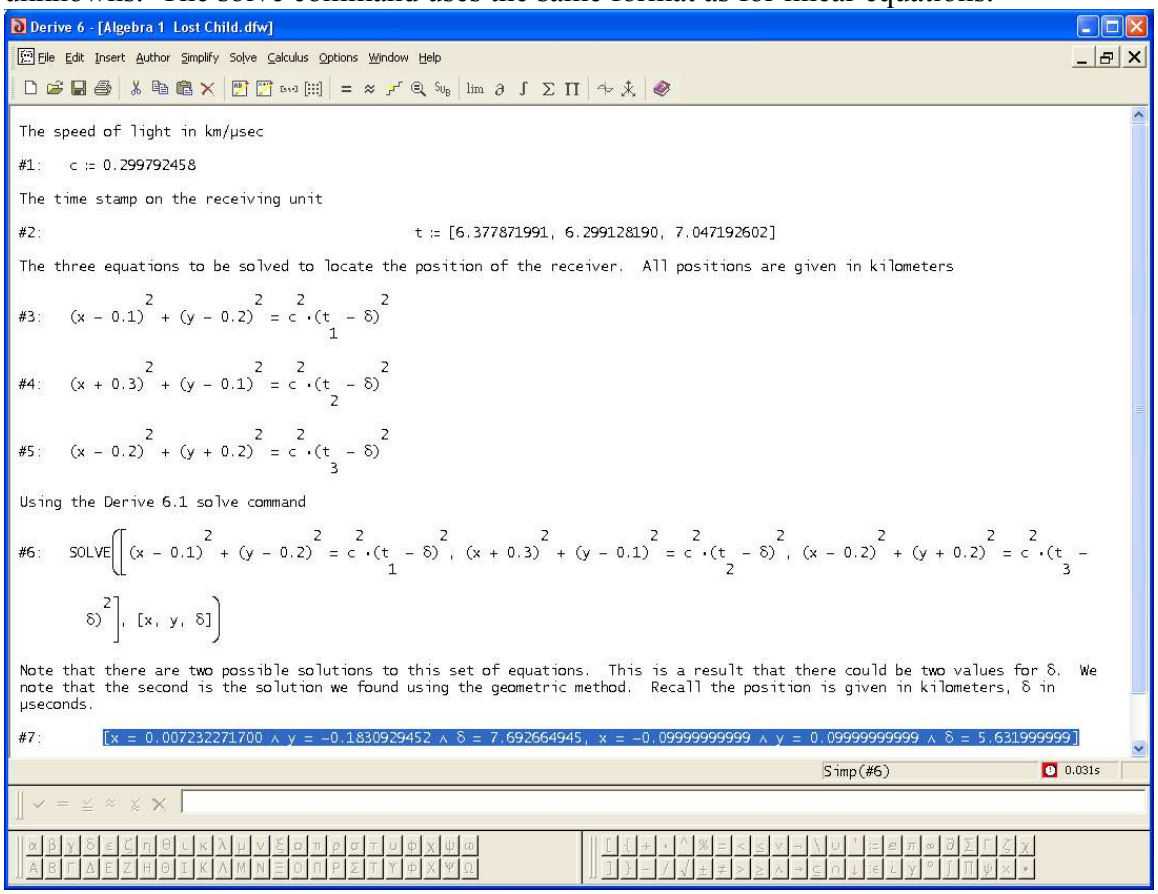
To help us out with the solution of our problem, we will use the non-linear equation solver that exists on our CAS. Yes, we are accepting a solution that uses a method that we do not know, and in mathematics, blind faith can be dangerous. On the other hand, it is the author's hope that you will pursue the study of such methods, as you learn more mathematics. Consider this tool as our means of whetting your appetite for more advanced study.

We will solve the same problem of the lost child that we considered above. Remember that our unknowns are x, y, and δ. We will use the value of

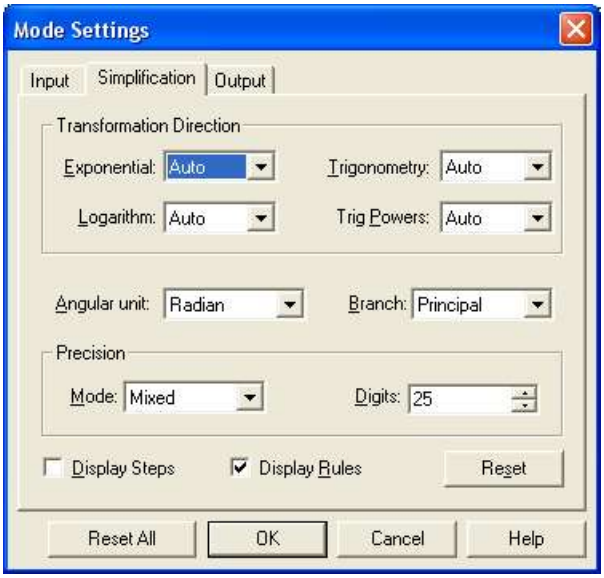
$$c = 0.299792458 \text{ km/}\mu\text{sec}$$

for the speed of light. All of our measurements will be in kilometers.

The following is the Derive[tm] 6.1 screen for solving the system of three equations in 3 unknowns. The solve command uses the same format as for linear equations.



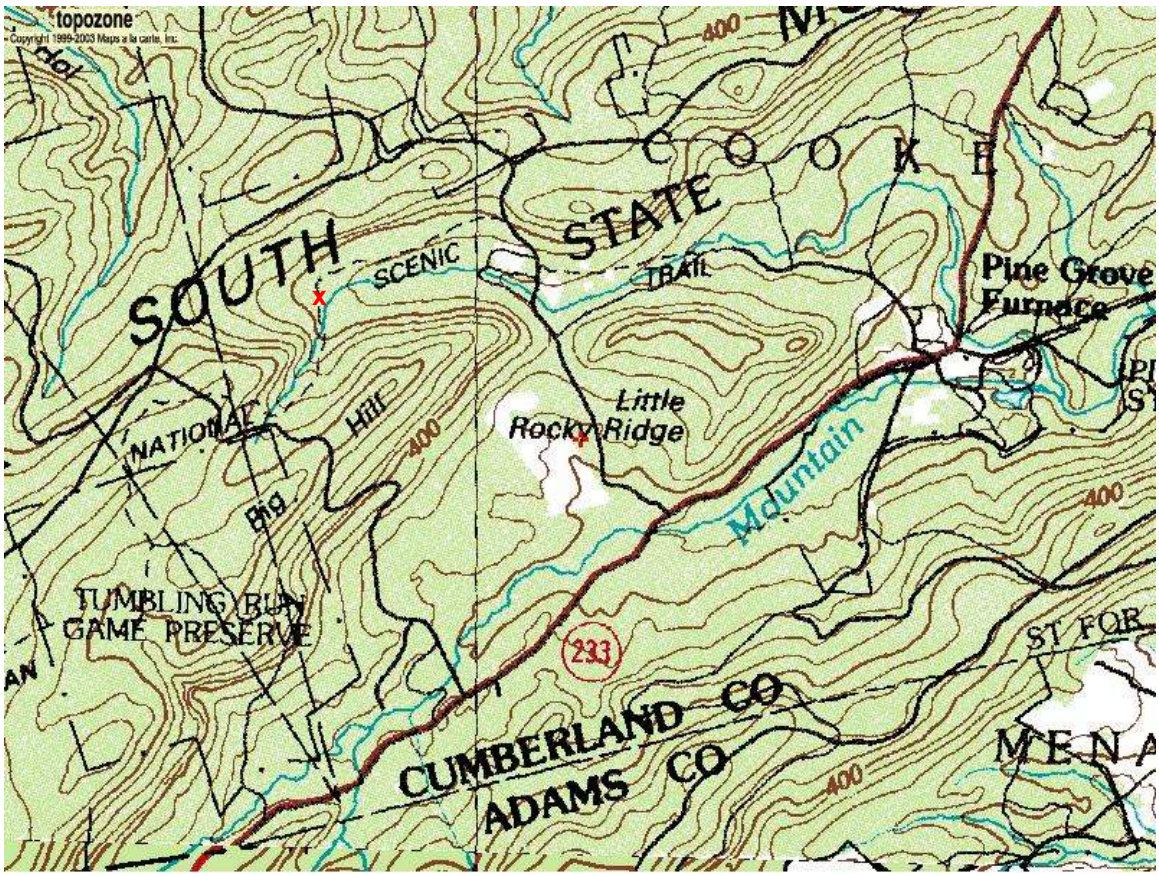
The mode settings (Options > Mode > Simplification) for this exercise were:



One last caution for this exercise: use the simplify button (=), not the approximate button (≈) after entering the solve command.

Creating An Exercise For Others to Solve

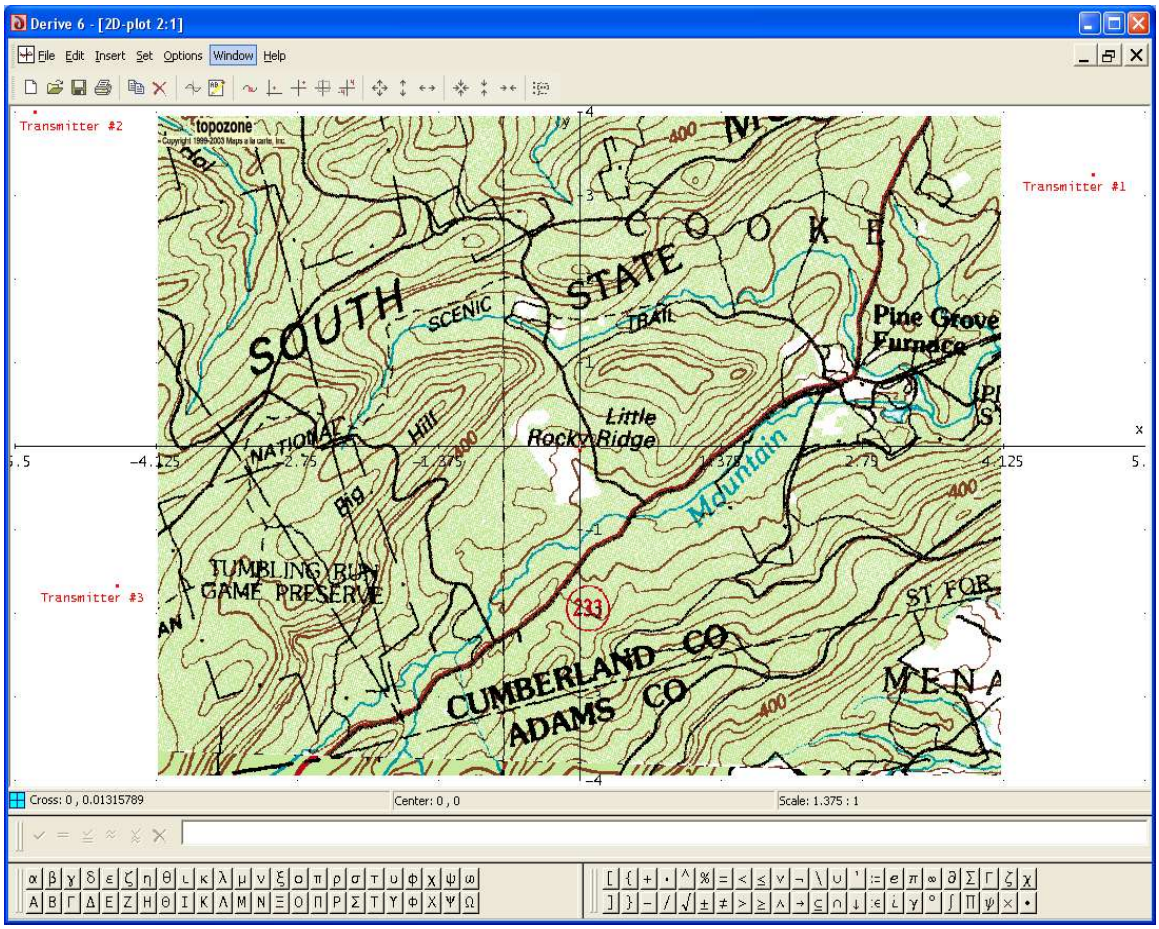
For this section we return to the situation described in the scenario. The hikers were on a trail in the deep woods. Below is a topographical map of the area where they were hiking. The map was obtained from **topozone**[tm] (http://www.topozone.com). The center of the map is located at GPS coordinates 40° 2.77'N 77° 22.28'W. The scale is: 1 km = 2 cm



A red 'x' just below the 'H' in 'SOUTH' locates the spot where the bones were found along the hiking trail.

We are going to use Derive^{tm}-6.1 to create a problem based on the scenario. Our first task is to create a Derive graphing screen that has the map in the background with the center of the map at the origin of the coordinate system. Follow these steps:

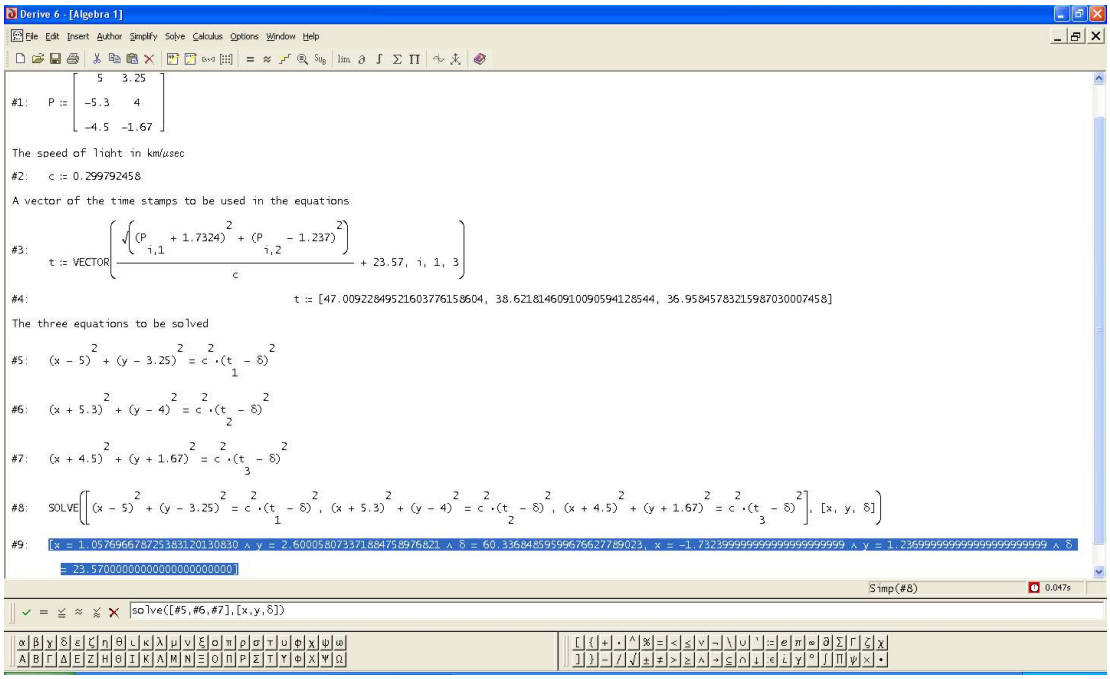
1. Open a Derive full screen graphing window
2. Using a metric ruler measure the length of each axis and choose a range for each axis (Set > Plot Range > Length/Center) so that 1unit = scale for 1 km. (in our case 1 unit = 2 cm)
3. If the map image is not a Bit Map image, create a Bit Map image of the map.
4. Go to Options > Display > Color > Picture. Click on the button to the right of right of the empty text box and browse for the Bit Map image of your map. Place its name in the text box. Choose the 'Center' option and click OK.
5. Using the pointer check the location of where the bones were found relative to the center of the coordinated system. In our case it is at (-1.734, 1.237)
6. Arbitrarily, choose three positions for the transmitter towers. They need not appear on the screen, but for this illustrative problem, the points chosen were (5, 3.25); (-5.3, 4); and (-4.5, -1.67). These plots were plotted and labeled as shown below.



NOTE: The map size has been reduced to fit the Derive screen onto this page. However, the scale has been preserved.

We now outline the procedure for constructing a problem to be solved by someone else.

1. Decide on a time discrepancy, δ, between the synchronized transmitter clocks and the GPS receiver. Let's choose $\delta = 23.57$ µsec. Don't publish this value, but it is needed to compute the time stamp to be used in the equations to be solved.
2. Calculate the times in µsec needed for the signals to reach the GPS receiver.
    a. Compute the distance in km from each transmitter to the (unpublished) position of the GPS receiver.
    b. Divide these distances by the speed of light in km/µsec.
3. Add δ to each of these times, these are the time stamps to be used in the equations.
4. Create the equations to be solved and check the result.



Once again there are two answers for the problem. The second is the one that you are searching for. You are now ready to publish the map and your problem giving the positions of the transmitters and the time stamps shown in the receiver.

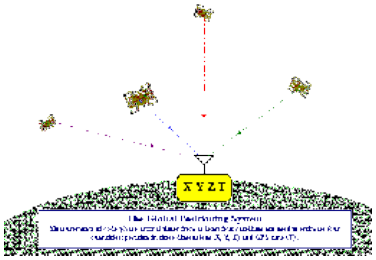
3-Dimensional GPS

Much of the material for this section was taken from the web site of Peter H. Dana of the Department of Geography of the University of Texas at Austin. The URL for this information is located at the University of Colorado.

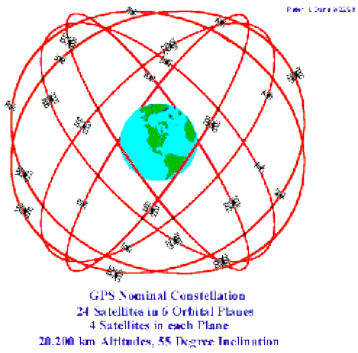
http://www.colorado.edu/geography/gcraft/notes/gps/gps_f.html

Having carefully worked through the two dimensional problem, the basic workings of the three dimensional GPS should be clear to us. Instead of being able to determine the x and y-coordinates of an object, we can determine the x, y, <u>and</u> z-coordinates, i.e. latitude, longitude, and altitude. Of course, our GPS receiver will have an unknown time discrepancy with the atomic clocks in the sending satellites, so we end up solving a system of four non-linear equations with four unknowns.

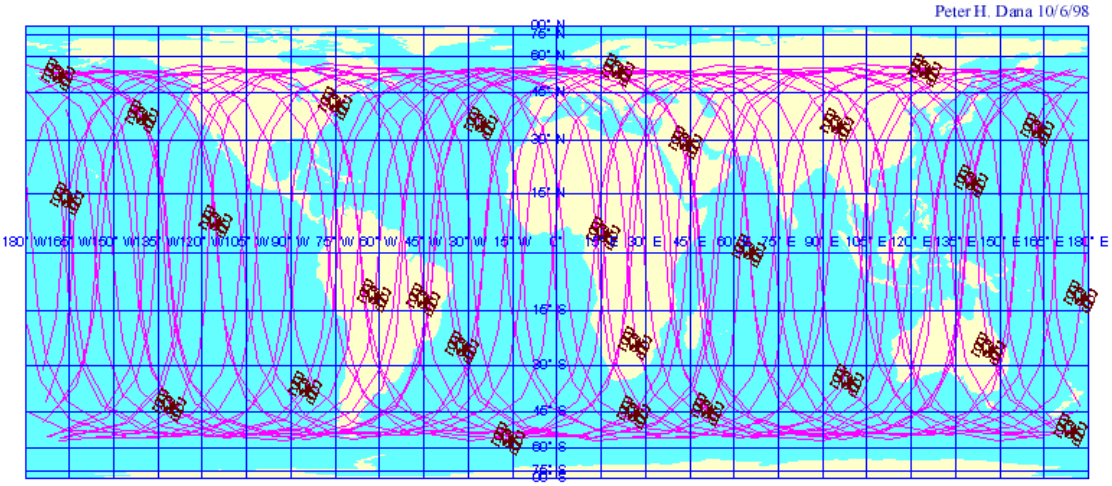
Contrary to what some may think, a GPS satellite knows nothing about the position of your GPS receiver. It knows only two facts: its own position in space and the exact time as sychronized by an atomic clock with the times of all of the other GPS satellites. Your GPS receiver receives this information from four satellites and calculates your distance from each of the satellites and the discrepancy between its clock and the satellite clocks. From this information, it can determine to within a certain numerical accuracy your position on the Earth. (See illustration below.)



There are 27 GPS satellites positioned in orbits that are 22,000 kilometers (13670.1662292 miles) above mean sea level, MSL. The satellites orbits are at an angle of 55° with the Earth's equator. We will assume that at MSL the Earth is a sphere of radius 3982.998 miles. This positioning is shown in the following illustration



The satellites are positioned so that at any moment, every spot on earth is "visible" by at least four of the satellites. Here is an illustration that depicts the satellites positions above the earth at one moment on September 29, 1998.



Global Positioning System Satellites and Orbits
for 27 Operational Satellites on September 29, 1998
Satellite Positions at 00:00:00 9/29/98 with 24 hours (2 orbits) of Ground Tracks to 00:00:00 9/30/98

3. To inspire you to learn more mathematics so that you can understand the mathematics behind the solutions found by the Computer Algebra System.

MATERIALS NEEDED

1. A Computer Algebra System with the capability of placing pictures on the coordinate system of the graphing tool, the ability to graph functions and equations having an undefined parameter, a slider bar, and the ability to give highly accurate approximate decimal solutions to systems of nonlinear equations.

2. A detailed map of a restricted area that, if possible, contains your school's location and other public buildings, monuments, streets, etc. in that location. An alternative is a topographical map of an area which is near your location.

3. A metric ruler or measuring tape.

ACTIVITY

Geometrically Solving the Problem

1. Using the topographical map given above that is not on the Derive screen, but has the scale of 2cm = 1km, arbitrarily pick a new location for the GPS receiver.

2. Place a coordinate system on the map with its origin at the center of the map. Place tick marks along the axes at distances of one km.

3. Place the three transmitters at positions having integer *xy*-coordinates. They need not be on the detailed portion of the map.

4. Measure the distance from each transmitter to the location of the receiver on the map.

5. Write the equations for circles with their centers at each of the transmitters and with the radius of each circle as a separate parameter.

6. Open the graphics screen and set the aspect ratio to 1:1 and set up slider bars for each of the radii of the circles. Be sure to include a range of values that will allow the circles to intersect.

7. Find approximate values for the position of the GPS receiver.

Creating A Positioning Problem To Be Solved

1. Using the map of your location, choose a prominent land mark (public building, monument, or street intersection) as the location for your receiver.

2. Place coordinate axes on your map and use the map scale to establish units for the coordinate system.

3. Choose three locations for your transmitters and a value in μseconds for δ, the time discrepancy between the receiver and the transmitters.

4. Follow the steps given above and construct your problem.

5. Before giving the problem to the class, check it by solving it yourself.

CORONER'S COMMENTS

Every coroner has had the experience of being called to supposed coroner's scenes only to find that the unearthed bones belong to an animal. The most common is the burial scene for a family pet. I had the experience of being called to a 'scene' by the State Police only to find that the bones were those of a beef cow that was butchered and the bones dumped off in a particularly bushy forested area. Because all of the meat had not been cleaned off of the bones, the smell was most unpleasant. Another time, I was called to the scene of a home remodeling. There was the fear that a disposal place for a mass murderer had been unearthed. It turned out that the former owner had had a small butcher shop on the premises and the bones were discarded pork bones. As noted, the experience of having to identify non human skeleton bones is common.

It is always important for the coroner to make a detailed description of the scene of a coroner's case. The GPS is a tool that can be of help in making such a description. This is more commonly the case when the scene is in a remote location and not near any prominent land marks. At present, the GPS is used more commonly in crime prevention, and location of individuals wearing GPS locators. A recent use is for individuals suffering from Alzheimer's Disease who may be prone to wondering off alone.